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PROBLEM SOLVERS

calculus of one variable

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CALCULUS OF ONE VARIABLE

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Problem Solvers

Edited by L. Marder

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Calculus of One Variable

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Calculus of One Variable

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Preface

This book provides a set of solved problems in the elementary methods of one-variable calculus. A full discussion of the analytical foundations of the calculus would clearly be out of place, and definitions are stated without the terminology and symbolism of analysis. This is particularly true of the notion of limit introduced in Chapter 1, where the ideas have been formulated in language more geometrical than analytical.

Although important theorems have been quoted without proof, a proper statement of the conditions under which they are valid has been given wherever possible, consistent with clarity of expression. For readers not following a course in analysis the full significance of these conditions may not be apparent. They do on the whole have geometrical interpretations, however, and readers should explore this aspect to gain a better understanding of the relevance of the conditions to their respective theorems.

When dealing with derivatives a choice has to be made between the notations dy/dx and $f'(x)$. I have chosen the latter for preference, and have considered it in the best interests of clarity to adhere to this notation throughout the book. Readers who have met both notations will judge their relative merits for themselves.

In the important topic centred around Taylor's theorem emphasis has been placed on errors and approximations, an aspect which is becoming increasingly prominent as more problems in advanced applications of calculus are solved numerically using a computer. Numerical estimation of integrals has been omitted however for reasons of space, although this too is important.

Whilst the applications of integration have been divided between geometrical and physical problems, the applications of the derivative given are mainly geometrical. Physical problems involving derivatives lead generally to differential equations, and techniques for solving these require a separate detailed study.

My grateful thanks are due to my wife, Ann, who has read this book in manuscript and in proof, and provided answers to the exercises. Her forbearance has made the preparation of the book a much easier task.

K.E.H.

Chapter 1
Limits

1.1 Definitions The function $f(x)$ is said to tend to l (have the limit l) as x tends to a, provided that the difference between $f(x)$ and l is smaller in magnitude than any prescribed positive real number for all values of x sufficiently close to, but different from a. We write $f(x) \to l$ as $x \to a$, or $\lim_{x \to a} f(x) = l$. The definition is illustrated by the following two problems.

Problem 1.1 Show that $\lim_{x \to 2} x^2 = 4$.

Solution. We wish to compare x^2 with 4. Thus

$$|x^2-4| = |(x-2)(x+2)| < |x-2|.10$$

provided $|x+2| < 10$ (i.e. $-12 < x < 8$). We may now make $|x^2-4|$ small by choosing x sufficiently close to 2. For example, $|x^2-4| < 10^{-6}$ provided that $|x^6-2|.10 < 10^{-6}$, i.e. provided that $|x-2| < 10^{-7}$. □

Problem 1.2 Show that $|x|/\sqrt{|x|} \to 0$ as $x \to 0$.

Solution. The definition excludes consideration of the point $x = 0$, and so the fact that the formula does not define a real number when $x = 0$ does not prevent the function having a limit there. Now

$$|x|/\sqrt{|x|} = \sqrt{|x|} \qquad \text{for } x \neq 0,$$

and so $|x|/\sqrt{|x|}$ can be made very close to zero provided x is very close to zero. For example, $|x|/\sqrt{|x|} < 10^{-10}$ provided $|x| < 10^{-20}$. □

Problem 1.3 Show that $|x|/x$ does not have a limit as $x \to 0$.

Solution. If $x > 0$ then $|x|/x = 1$. If $x < 0$ then $|x|/x = -1$. No number l can be found which is very close to both 1 and -1. □

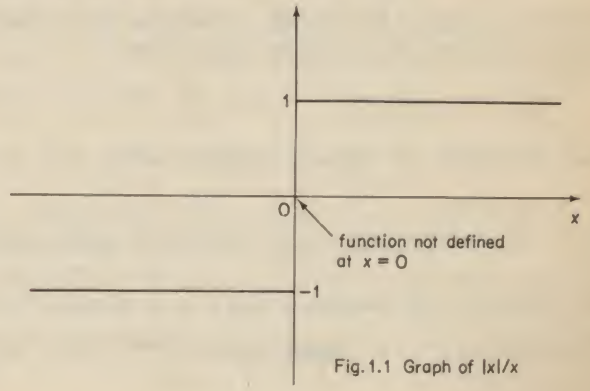

Fig. 1.1 Graph of $|x|/x$

Further definitions (i) The function $f(x)$ is said to tend to infinity as x tends to a provided that the values of $f(x)$ are larger than any prescribed real number for all values of x sufficiently close to a. We write $f(x) \to \infty$ as $x \to a$.

(ii) The function $f(x)$ is said to tend to infinity as x tends to infinity provided that the values of $f(x)$ are larger than any prescribed real number for all values of x sufficiently large. We write $f(x) \to \infty$ as $x \to \infty$.

(iii) The function $f(x)$ is said to tend to the limit l as x tends to infinity provided that the difference between $f(x)$ and l is smaller in magnitude that any prescribed positive real number for all sufficiently large values of x. We write $f(x) \to l$ as $x \to \infty$ or $\lim_{x\to\infty} f(x) = l$.

Similar definitions may be formulated with minus infinity replacing infinity.

Problem 1.4 Show that $1/x$ tends to zero as x tends to infinity.

Solution. When x is very large $1/x$ is very close to zero. For example, for all values of $x > 10^{12}$ we have $0 < 1/x < 10^{-12}$. □

Continuity The function $f(x)$ is said to be continuous at $x = a$ provided that $\lim_{x\to a} f(x) = f(a)$.

Theorem 1.1 If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then

(i) $\lim_{x\to a} (f(x)+g(x)) = l+m$,

(ii) $\lim_{x\to a} f(x).g(x) = l.m$,

(iii) $\lim_{x\to a} f(x)/g(x) = l/m$ provided $m \neq 0$.

Similar results are true with the limit as x tends to a replaced by the limit as x tends to infinity.

There are many types of limits used in calculus, and we shall consider examples from three important classes of functions.

1.2 Rational Functions A rational function is the ratio of two polynomials.

Problem 1.5 Show that a polynomial function is continuous for all real values of x.

Solution. For any real number a, $\lim_{x\to a} x = a$. Using Th. 1.1(ii) $n-1$ times, we obtain $\lim_{x\to a} x^n = a^n$. Again applying Th. 1.1(ii), with $f(x) = p$ (constant) and $g(x) = x^n$, we obtain $\lim_{x\to a} px^n = pa^n$. If

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0,$$

then repeated application of Th. 1.1(i) yields

$$\lim_{x\to a} P(x) = p_n a^n + p_{n-1} a^{n-1} + \ldots + p_1 a + p_0 = P(a),$$

and so $P(x)$ is continuous at a. □

Problem 1.6 Show that any rational function is continuous for all values of x for which the denominator is non-zero.

Solution. Let $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials. Suppose $Q(a) \neq 0$. By Problem 1.5,

$$\lim_{x\to a} P(x) = P(a) \quad \text{and} \quad \lim_{x\to a} Q(x) = Q(a).$$

Thus by Th. 1.1(iii), since $Q(a) \neq 0$, $\lim_{x\to a} P(x)/Q(x) = P(a)/Q(a) = R(a)$. □

Problem 1.7 Show that

$$\lim_{x\to\infty} \frac{2x^2+3x+4}{4x^3+x-7} = 0.$$

Solution.

$$\frac{2x^2+3x+4}{4x^3+x-7} = \left(\frac{2}{x}+\frac{3}{x^2}+\frac{4}{x^3}\right) \Big/ \left(4+\frac{1}{x^2}-\frac{7}{x^3}\right) = \frac{P(x)}{Q(x)}.$$

Repeated application of Th. 1.1 in conjunction with Problem 1.4 shows that $\lim_{x\to\infty} P(x) = 0$, $\lim_{x\to\infty} Q(x) = 4$. Thus $\lim_{x\to\infty} [P(x)/Q(x)] = 0/4 = 0$. □

Problem 1.8 Evaluate

$$\lim_{x\to\infty} \frac{4x^3-3x^2+2}{2x^3-4x^2+x-1}.$$

Solution.

$$\frac{4x^3-3x^2+2}{2x^3-4x^2+x-1} = \left(4-\frac{3}{x}+\frac{2}{x^3}\right) \Big/ \left(2-\frac{4}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right) = \frac{P(x)}{Q(x)}.$$

As in Problem 1.7 we have $\lim_{x\to\infty} P(x) = 4$, $\lim_{x\to\infty} Q(x) = 2$. Thus

$$\lim_{x\to\infty} [P(x)/Q(x)] = 4/2 = 2.$$ □

Problem 1.9 Show that if $f(x) \to 0$ as $x \to \infty$, and if $f(x) > 0$ for all sufficiently large values of x, then $1/f(x) \to \infty$ as $x \to \infty$.

Solution. Since $f(x) \to 0$ as $x \to \infty$, $f(x)$ is very small when x is very large. Thus $1/f(x)$ is very large when x is very large. For example, we can find a value of t so that for all $x > t$, $0 < f(x) < 10^{-6}$. Thus for all $x > t$, $1/f(x) > 10^6$.

Problem 1.10 Show that

$$\frac{4x^3+x-7}{2x^2+3x+4} \to \infty \qquad \text{as } x \to \infty.$$

Solution. By Problem 1.7, $(2x^2+3x+4)/(4x^3+x-7) \to 0$ as $x \to \infty$. Thus by Problem 1.9, $(4x^3+x-7)/(2x^2+3x+4) \to \infty$ as $x \to \infty$, since the function is certainly positive for all values of $x > 7$. □

In Problem 1.6 we investigated the limit of a rational function in cases where the denominator is non-zero. In the following problem the denominator vanishes for the value of x under discussion.

Problem 1.11 Evaluate

$$\lim_{x\to 2} \frac{3x^3-2x^2-7x-2}{x^2+2x-8}.$$

Solution. Write $P(x) = 3x^3-2x^2-7x-2$, $Q(x) = x^2+2x-8$. Then $P(2) = Q(2) = 0$. We can therefore factorize $P(x)$ and $Q(x)$ as follows:

$$P(x) = (x-2)(3x^2+4x+1), \qquad Q(x) = (x-2)(x+4).$$

Thus for $x \neq 2$,

$$\frac{P(x)}{Q(x)} = \frac{3x^2+4x+1}{x+4}.$$

In evaluating the limit we are concerned with values of x except $x = 2$, and so, using Problem 1.5, and Th. 1.1(iii),

$$\lim_{x\to 2} \frac{3x^3-2x^2-7x-2}{x^2+2x-8} = \lim_{x\to 2} \frac{3x^2+4x+1}{x+4} = \frac{21}{6} = \frac{7}{2}.$$

In cases where the denominator vanishes at $x = a$ and the numerator does not, difficulties arise over variation in sign of the denominator in the region of $x = a$, and so Problem 1.9 cannot be applied directly. A discussion of one-sided limits will help to describe the behaviour of the function in such cases (see § 3.4).

1.3 Trigonometric Functions

Problem 1.12 Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Solution. Elementary trigonometry books use a geometrical definition of the sine function and give a geometrical proof for the limit. We shall use the definition of $\sin x$ as a series and give an algebraic proof.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

Since there are infinitely many terms on the right-hand side of this equation we cannot apply Th. 1.1(i) to deduce the result. We derive an inequality involving a geometric series.

$$1 - \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{8} - \dots < \frac{\sin x}{x} < 1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{8} + \dots$$

$$\frac{1-x^2}{1-\frac{1}{2}x^2} < \frac{\sin x}{x} < \frac{1}{1-\frac{1}{2}x^2}.$$

But

$$\lim_{x\to 0} \frac{1-x^2}{1-\frac{1}{2}x^2} = \lim_{x\to 0} \frac{1}{1-\frac{1}{2}x^2} = 1$$

and so $\lim_{x\to 0} (\sin x/x) = 1$. □

Many problems involving trigonometric limits can be reduced to that of Problem 1.12 by a change of variable. It is therefore convenient to quote a theorem at this point which enables us to do this.

Theorem 1.2 Suppose $\lim_{x\to a} f(x) = l$ and $\lim_{t\to b} g(t) = a$. Then

$$\lim_{t\to b} f(g(t)) = l.$$

Similar results may be formulated in cases where a, b or l is $\pm\infty$. We shall use such results where appropriate.

Problem 1.13 Evaluate $\lim_{x\to 0} (\sin^2 2x/x^2)$

Solution. Write $g(t) = \frac{1}{2}t$, then $\lim_{t\to 0} g(t) = 0$. Thus

$$\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = \lim_{t\to 0} \frac{\sin^2 t}{(\frac{1}{2}t)^2} = \lim_{t\to 0} 4\left(\frac{\sin t}{t}\right)^2 = 4.$$ □

With practice specific reference to the function $g(t)$ may be dropped and the substitution made as in the following problems.

Problem 1.14 Evaluate

$$\lim_{x\to 1} \frac{\sin(x^2-1)}{(x-1)}$$

Solution.

$$\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x\to 1} (x+1)\frac{\sin(x^2-1)}{x^2-1}$$

$$= \lim_{x\to 1}(x+1)\lim_{x\to 1}\frac{\sin(x^2-1)}{x^2-1}$$

$$= 2\lim_{t\to 0}\frac{\sin t}{t} = 2, \quad \text{(writing } t = x^2-1\text{)}. \ \square$$

Problem 1.15 Evaluate $\lim_{x\to 0}(\sin mx/\sin nx)$.

Solution.

$$\frac{\sin mx}{\sin nx} = \frac{\sin mx}{mx}\frac{nx}{\sin nx}\frac{mx}{nx} = \frac{m}{n}\frac{\sin mx}{mx}\frac{nx}{\sin nx}.$$

Thus

$$\lim_{x\to 0}\frac{\sin mx}{\sin nx} = \frac{m}{n}\lim_{x\to 0}\frac{\sin mx}{mx}\lim_{x\to 0}\frac{nx}{\sin nx} = \frac{m}{n}.1.1 = \frac{m}{n} \qquad \square$$

Problem 1.16 Evaluate $\lim_{x\to 0}[(1-\cos x)/x^2]$.

Solution. Again the problem is reduced to one like Problem 1.12.

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2\frac{1}{2}x}{x^2} = \frac{1}{2}\left(\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}\right)^2 \to \frac{1}{2} \qquad \text{as } x\to 0. \qquad \square$$

1.4 Exponential Functions The usual definition of the exponential function is by means of a series:

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots .$$

The theorem that $\exp(x+y) = \exp x.\exp y$ leads to the use of indices, and we shall sometimes write e^x for $\exp x$.

Theorem 1.3 $$\lim_{x\to\infty}\left(1+\frac{t}{x}\right)^x = e^t.$$

Problem 1.17 Show that, for any positive integer n, $e^x/x^n \to \infty$ as $x\to\infty$. (i.e. the exponential function increases more rapidly than any power of x.)

Solution.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \dots .$$

Since all the terms are positive when $x > 0$, we have $e^x > x^{n+1}/(n+1)!$. Thus

$$\frac{e^x}{x^n} > \frac{1}{x^n}\frac{x^{n+1}}{(n+1)!} = \frac{x}{(n+1)!} \to \infty \qquad \text{as } x\to\infty,$$

and so $e^x/x^n \to \infty$ as $x\to\infty$. (A slight modification allows the positive integer n to be replaced by any positive real number.) $\square$

The logarithmic function can be defined in several ways, including the definition as the inverse of the exponential function, i.e. if $y = e^x$, we define the logarithmic function by the equation $x = \log_e y$. Throughout this book we shall use logarithms to base e and not to base 10. The subscript e will therefore be omitted, and $\log_e x$ will be denoted by the standard notation $\ln x$.

Problem 1.18 For any positive real number a, $\lim_{x\to\infty}\ln x/x^a = 0$.

Solution. We use Th. 1.2 and the previous problem, with the modification indicated at the end. Let $t = \ln x$, then

$$\lim_{x\to\infty}\frac{\ln x}{x^a} = \lim_{t\to\infty}\frac{t}{(e^t)^a} = \lim_{t\to\infty}\frac{t}{e^{at}} = \lim_{y\to\infty}\frac{y}{ae^y} = 0,$$

using Problem 1.17 and a reciprocal version of Problem 1.9. $\square$

The next two problems are once again cases which reduce algebraically to standard limits.

Problem 1.19 Evaluate $\lim_{x\to\infty}[(x+1)/(x-1)]^x$.

Solution.

$$\left(\frac{x+1}{x-1}\right)^x = \frac{(1+1/x)^x}{(1-1/x)^x} \to \frac{e}{e^{-1}} = e^2 \qquad \text{as } x\to\infty,$$

using Th. 1.1(iii) and Th. 1.3. $\square$

Problem 1.20 Evaluate $\lim_{x\to\infty}[(\cosh^2 x + \sinh^2 x)/x^4]$.

Solution. The hyperbolic functions $\cosh x$ and $\sinh x$ are defined by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

Thus we have

$$\frac{\cosh^2 x + \sinh^2 x}{x^4} = \frac{(e^x+e^{-x})^2 + (e^x-e^{-x})^2}{4x^4}$$

$$= \frac{1}{4}\left[\left(\frac{e^x}{x^2}+\frac{e^{-x}}{x^2}\right)^2 + \left(\frac{e^x}{x^2}-\frac{e^{-x}}{x^2}\right)^2\right]$$

Now $e^x/x^2 \to \infty$ as $x\to\infty$ and $e^{-x}/x^2 \to 0$ as $x\to\infty$. Thus we obtain the result that

$$\frac{\cosh^2 x + \sinh^2 x}{x^4} \to \infty \qquad \text{as } x\to\infty. \qquad \square$$

Problem 1.21 Evaluate $\lim_{x\to 0}[(e^x-1)/x]$.

Solution. We use a technique similar to that of Problem 1.12.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

and so for $x \neq 0$,

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots .$$

We therefore have

$$1 - \frac{|x|}{2} - \frac{|x|^2}{4} - \frac{|x|^3}{8} - \dots < \frac{e^x - 1}{x} < 1 + \frac{|x|}{2} + \frac{|x|^2}{4} + \frac{|x|^3}{8} + \dots$$

$$\frac{1-|x|}{1-\frac{1}{2}|x|} < \frac{e^x - 1}{x} < \frac{1}{1-\frac{1}{2}|x|}$$

Now $$\lim_{x\to 0} \frac{1-|x|}{1-\frac{1}{2}|x|} = \lim_{x\to 0} \frac{1}{1-\frac{1}{2}|x|} = 1$$

and so $$\lim_{x\to 0} \frac{e^x-1}{x} = 1.$$ □

EXERCISES

1. $P(x)$ is a polynomial of degree m, and $Q(x)$ is a polynomial of degree n. Investigate the manner in which $\lim_{x\to\infty}[P(x)/Q(x)]$ depends upon m and n.

2. Evaluate $\lim_{x\to 3} \dfrac{x^3+3x^2-19x+3}{4x^4-12x^3-x^2-x-6}$.

3. Evaluate $\lim_{x\to -2} \dfrac{x^3+x^2-8x-12}{x^2+3x+2}$.

4. Evaluate $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

5. Evaluate $\lim_{x\to 0} \dfrac{2\sin^2 3x}{x^2}$.

6. Evaluate $\lim_{x\to 0} \dfrac{e^x-1-x}{x^2}$.

Chapter 2

Differentiation

2.1 Definitions and Elementary Rules The function $f(x)$ is said to be differentiable at $x = a$ if the quotient

$$\frac{f(a+h)-f(a)}{h}$$

has a limit as $h \to 0$. If the limit does exist, then that number is called the differential coefficient of $f(x)$ at a, denoted by $f'(a)$. The following rules follow from the properties of limits without difficulty.

(i) *Linearity*: If $f(x)$ and $g(x)$ are both differentiable at $x = a$, and if p and q are constants, then the function $(pf+qg)(x)$ is differentiable at $x = a$, and

$$(pf+qg)'(a) = pf'(a)+qg'(a).$$

(ii) *Product Rule*: If $f(x)$ and $g(x)$ are both differentiable at $x = a$ then the function $(fg)(x)$ is differentiable at $x = a$ and

$$(fg)'(a) = f'(a)\,g(a)+f(a)\,g'(a).$$

(iii) *Quotient Rule*: If $f(x)$ and $g(x)$ are both differentiable at $x = a$, and also $g(a) \neq 0$, then the function $(f/g)(x)$ is differentiable at $x = a$ and

$$(f/g)'(a) = \frac{g(a)\,f'(a)-f(a)\,g'(a)}{(g(a))^2}.$$

2.2 Standard Results Many of the elementary functions encountered in the calculus are differentiable at most values of x for which they are defined. In these circumstances we can consider a new function whose value at $x = a$ is $f'(a)$. We call this the derived function or derivative of $f(x)$, and we use the notation $f'(x)$ for this function. Three basic examples are dealt with in this section, and from them many other functions can be differentiated by means of the rules.

Problem 2.1 To find the derivative of $f(x) = x^n$, where n is a positive integer.

Solution. $$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n-x^n}{h}$$

$$= \frac{1}{h}(x^n+nx^{n-1}h+\tfrac{1}{2}n(n-1)x^{n-2}h^2+\dots+h^n-x^n)$$

(using the binomial theorem)

$$= nx^{n-1} + \tfrac{1}{2}n(n-1)x^{n-2}h + \ldots + h^{n-1} - x^n)$$
$$\to nx^{n-1} \qquad \text{as } h \to 0,$$

since each of the $n-1$ terms after the first contains a power of h, and so tends to zero as $h \to 0$. Thus $f(x)$ is differentiable for any value of x, and $f'(x) = nx^{n-1}$. □

Problem 2.2 To find the derivative of $f(x) = \sin x$.

Solution.

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h} = \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$= \cos(x+\tfrac{1}{2}h)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \to \cos x \qquad \text{as} \quad h \to 0,$$

using Problem 1.12 modified by means of Th. 1.2. Thus $f(x)$ is differentiable for any value of x and $f'(x) = \cos x$. □

Problem 2.3 To find the derivative of $f(x) = e^x$.

Solution. Using Problem 1.21,

$$\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^x}{h} = \frac{e^x e^h - e^x}{h} = \left(\frac{e^h-1}{h}\right)e^x \to e^x \text{ as } h \to 0.$$

Thus $f(x)$ is differentiable for all values of x and $f'(x) = e^x$. □

2.3 The Chain Rule

Theorem 2.1 Suppose that $f(x)$ is a function which is differentiable at $x = a$, and that $g(t)$ is differentiable at $t = f(a)$. Then the composite function $g \circ f(x) = g(f(x))$ is differentiable at $x = a$ and

$$g \circ f'(a) = g'(f(a))\, f'(a).$$

Problem 2.4 Find the derivative of $h(x) = \sin(x^2)$.

Solution. Write $f(x) = x^2, g(t) = \sin t$. Then $f'(x) = 2x$ and $g'(t) = \cos t$. Thus by Th. 2.1,

$$h'(x) = g'(f(x))\, f'(x) = \cos(x^2).2x. \qquad \square$$

Problem 2.5 Find the derivative of $k(x) = e^{\sin(x^2)}$.

Solution. Th. 2.1 is applied twice in this example. Write $l(s) = e^s$ and $h(x) = \sin(x^2)$. Then by Th. 2.1,

$$k'(x) = l'(h(x))\,h'(x) = e^{\sin(x^2)}.\,2x\cos(x^2), \qquad \text{using Problem 2.4.}$$

Alternatively, without explicit reference to Problem 2.4 we can write

$l(s) = e^s, h(x) = \sin(x^2), g(t) = \sin t, f(x) = x^2$. Applying Th. 2.1 twice then gives

$$k'(x) = l'(h(x))h'(x) = l'(h(x))\,g'(f(x))\,f'(x)$$
$$= e^{h(x)}\cos(f(x)).\,2x = e^{\sin(x^2)}\cos(x^2).\,2x.$$

With practice specific reference to the intermediate functions $l(s), h(x), g(t), f(x)$ would be dropped, and one would proceed as follows:

To differentiate $k(x) = e^{\sin(x^2)}$

Mental manipulations (1, 2, 3)

$$k'(x) = e^{\sin(x^2)}\cos(x^2)\,2x$$

1. The derivative of e^* is e^*.
2. The derivative of $\sin *$ is $\cos *$.
3. The derivative of x^2 is $2x$. □

2.4 Applying the Rules of Differentiation Application of the rules for linearity, products, quotients and the chain rule yields further standard results from those in § 2.2.

Problem 2.6 Find the derivative of $f(x) = \cosh x$.

Solution. $f(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$. Thus $f'(x) = \frac{1}{2}e^x + \frac{1}{2}(e^{-x})(-1)$ (using linearity and the chain rule applied to e^{-x}); i.e.

$$f'(x) = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x} = \sinh x. \qquad \square$$

Problem 2.7 Find the derivative of $f(x) = \cos x$.

Solution. $f(x) = \cos x = \sin(\frac{1}{2}\pi - x)$. Applying the chain rule gives $f'(x) = \cos(\frac{1}{2}\pi - x)(-1)$, since the derivative of $(\frac{1}{2}\pi - x)$ is -1. Thus $f'(x) = -\sin x$. □

Problem 2.8 Find the derivative of $f(x) = \tan x$.

Solution. Since $\tan x$ is not defined if x is an odd multiple of $\frac{1}{2}\pi$, we may consider only values of x for which $\cos x \neq 0$. Applying the quotient rule we have, since $f(x) = \tan x = \sin x/\cos x$,

$$f'(x) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

□

Proficiency in differentiation of functions defined by means of complicated formulae can be achieved only by systematic practice, rather than by studying a long list of worked examples. We include therefore just one complicated example in this section.

Problem 2.9 Find the derivative of $f(x) = \cosh(e^{-x\cos^2 x})$.

Solution. In applying the procedure outlined at the end of Problem 2.5,

we use

1. The derivative of $\cosh *$ is $\sinh *$.
2. The derivative of e^* is e^*.
3. The derivative of $*^2$ is $2*$.
4. The derivative of $\cos x$ is $-\sin x$.
5. The derivative of x is 1.
6. The product rule to differentiate $x\cos^2 x$.

We then have

$$f(x) = \cosh(e^{-x\cos^2 x})$$

(1) (2) (5) (3) (4)

$$f'(x) = \sinh(e^{-x\cos^2 x})e^{-x\cos^2 x}\{-1\cos^2 x - x[2\cos x(-\sin x)]\}$$

$$= \sinh(e^{-x\cos^2 x})e^{-x\cos^2 x}(2x\cos x\sin x - \cos^2 x). \quad \square$$

In Problems 2.4–2.9 we have been able to find the derivative of the functions by applying the rules of differentiation together with knowledge of the derivatives of a few standard functions. The complicated functions we have dealt with have all been defined by single formulae and we have not needed to use the limiting procedure of § 2.2. The following is an example of a function not defined by a single formula, and to find the derivative at some points a we shall have to revert to the definition of $f'(a)$.

Problem 2.10 Investigate differentiability of the function $f(x)$ defined by

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leqslant x \leqslant 1 \\ x^3 - x + 1 & \text{if } x > 1. \end{cases}$$

Solution. The numbers $x = 0$ and $x = 1$ need to be considered separately. For other values of x, however, the rules can be applied to give

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 2x & \text{if } 0 < x < 1 \\ 3x^2 - 1 & \text{if } x > 1. \end{cases}$$

We now consider $x = 0$.

$$\frac{f(0+h)-f(0)}{h} = \begin{cases} \dfrac{-h-0}{h} = -1 & \text{if } h < 0 \\ \dfrac{(0+h)^2-0}{h} = h & \text{if } 0 < h < 1. \end{cases}$$

Thus $[f(0+h)-f(0)]/h$ does not have a limit as $h \to 0$, since it is -1 if h

is close to zero and negative, whereas it is close to 0 if h is close to zero and positive. Thus $f(x)$ is not differentiable at $x = 0$. (It is, however, continuous at $x = 0$.)

We finally consider $x = 1$.

$$\frac{f(1+h)-f(1)}{h} = \begin{cases} \dfrac{(1+h)^2-1}{h} = 2+h & \text{if } -1 < h < 0 \\ \dfrac{(1+h)^3-(1+h)+1-1}{h} = 2+3h+h^2 & \text{if } h > 0. \end{cases}$$

Thus

$$\frac{f(1+h)-f(1)}{h} \to 2 \qquad \text{as } h \to 0$$

and so $f'(1) = 2$. The graph of $f(x)$ is sketched in Fig. 2.1. $\square$

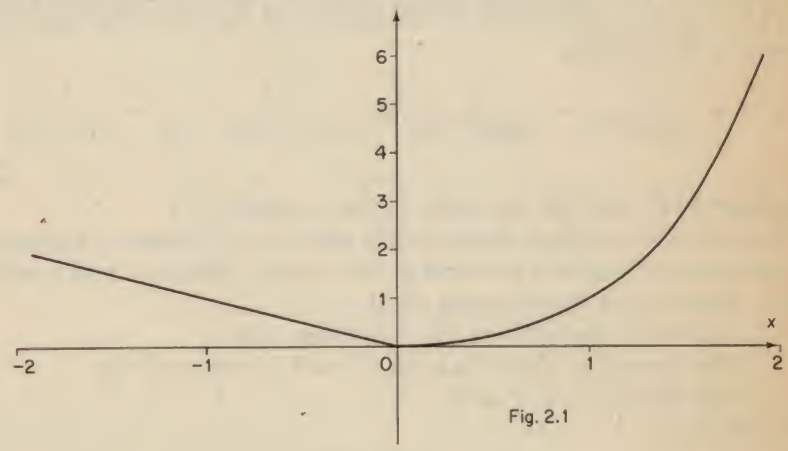

Fig. 2.1

2.5 Inverse Functions

***Theorem* 2.2** Suppose that the function $f(y)$ is strictly increasing (or strictly decreasing) for $c < y < d$, that $f(y)$ is differentiable at a, where $c < a < d$, and that $f'(a) \neq 0$. Then the inverse function $g(x)$ is differentiable at $f(a)$ and

$$g'(f(a)) = 1/f'(a).$$

Problem 2.11 Find the derivative of the function $g(x) = \sqrt{x}$. ($\sqrt{x}$ denotes the positive square root of x.)

Solution. We apply Th. 2.2 to the function $f(y) = y^2$. This function is strictly increasing for positive values of y, and the function $g(x)$ is the

inverse of $f(y)$ for $y > 0$. If $x = f(a)$, then $x = a^2$ so that $a = \sqrt{x}$. Applying Th. 2.2 therefore gives

$$g'(x) = \frac{1}{f'(\sqrt{x})} = \frac{1}{2\sqrt{x}}, \qquad \text{since } f'(y) = 2y. \qquad \square$$

Problem 2.12 Find the derivative of $g(x) = \ln x$.

Solution. $g(x) = \ln x$ is defined for positive values of x and is the inverse function of $f(y) = e^y$. If $x = e^a$ then $a = \ln x$, and so, applying Th. 2.2, we obtain

$$g'(x) = \frac{1}{f'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x},$$

since $f'(y) = e^y$. $\square$

Problem 2.13 Find the derivative of $g(x) = \sin^{-1}x$.

Solution. The function $f(y) = \sin y$ is strictly increasing for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$, and $\sin^{-1}x$ is the inverse of $f(y)$ over this interval. Applying Th. 2.2 gives

$$g'(x) = \frac{1}{f'(\sin^{-1}x)} = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{(1-\sin^2(\sin^{-1}x))}} = \frac{1}{\sqrt{(1-x^2)}}.$$

$\square$

Problem 2.14 Find the derivative of $f(x) = \ln(\sin^{-1}x^3)$.

Solution. This problem illustrates the methods of Problem 2.9 applied to the inverse function introduced in this section. The chain rule is used twice together with the following results:

1. The derivative of $\ln *$ is $1/*$ (Problem 2.12).
2. The derivative of $\sin^{-1} *$ is $1/\sqrt{(1-*^2)}$ (Problem 2.13).
3. The derivative of x^3 is $3x^2$.

We then have

$$f(x) = \ln(\sin^{-1}x^3)$$

$$f'(x) = \frac{1}{\sin^{-1}x^3}\frac{1}{\sqrt{(1-(x^3)^2)}}3x^2$$

$$= \frac{3x^2}{\sin^{-1}x^3\sqrt{(1-x^6)}}. \qquad \square$$

In the formula for $f'(x)$, any non-zero value of x satisfying $-1 < x < 1$ may be substituted to give a real number.. But in the formula for $f(x)$ we may only substitute values of x satisfying $0 < x \leqslant 1$, since $\ln y$ is not defined for $y \leqslant 0$. This point should be considered when complicated formulae occur. Formulae such as $\ln\ln\sin x$ do not give rise to real numbers for any substituted value of x, and so do not define functions at

all. Nevertheless, the rules of differentiation can be mechanically applied to give formulae which are sometimes meaningful for certain values of x.

2.6 Implicit Functions All the functions we have encountered so far have been defined explicitly by means of a formula involving the variable x. However, this may not always be so, and we may have a function $y = y(x)$ defined implicitly by an equation $f(x, y) = 0$. It may or may not be possible to solve this equation so as to express y explicitly in terms of x, but in either case it is often easier to work with the implicit equation in order to find $y'(x)$.

Problem 2.15 The function $y = y(x)$ satisfies the equation $x^2 + y^2 = 1$. Find $y'(x)$.

Solution. In this case we can solve the equation explicitly to obtain $y = \pm\sqrt{(1-x^2)}$. It will be seen that the equation $x^2 + y^2 = 1$ associates more than one value of y with most given values of x (this is shown on

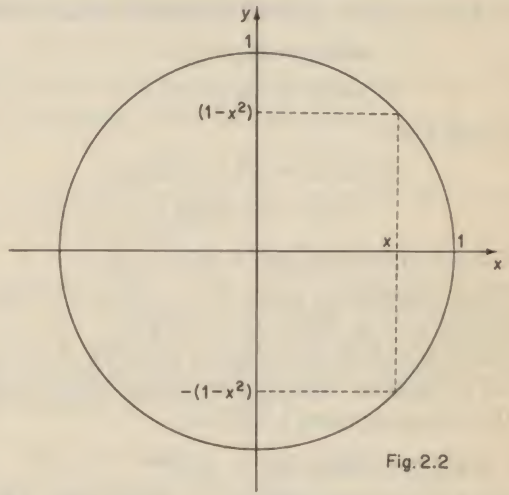

Fig. 2.2

Fig. 2.2). Thus there are many functions $y = y(x)$ which satisfy the equation, of which two may be defined explicitly by the formulae

$$y(x) = \sqrt{(1-x^2)}, \qquad y(x) = -\sqrt{(1-x^2)}.$$

Let us choose the first and compare two ways of finding $y'(x)$. Using the formula $y(x) = \sqrt{(1-x^2)}$ we obtain

$$y'(x) = -x/\sqrt{(1-x^2)}$$

by using the chain rule. The chain rule can also be applied to differentiate

y^2 in the implicit equation $x^2+y^2 = 1$, and we obtain

$$2x+2y\,y'(x) = 0. \text{ Hence } y'(x) = -x/y = -x/\sqrt{(1-x^2)}. \qquad \square$$

Problem 2.16 Find $y'(x)$ when $y = 1+x^2e^y$.

Solution. We use the product rule and the chain rule to obtain

$$y'(x) = 2x\,e^y+x^2e^y\,y'(x).$$

This equation becomes

$$y'(x) = \frac{2xe^y}{1-x^2e^y} = \frac{2(y-1)}{(2-y)x},$$

using the implicit equation to eliminate e^y. Note that in contrast to the previous example one cannot obtain an explicit formula for y and the expression for $y'(x)$ involves y as well as x. $\square$

Problem 2.17 Find $y'(x)$ when $2^y = x^2y$.

Solution. This problem illustrates a general method which can be applied to equations containing indices and products. We take logarithms of both sides to form another implicit equation. Thus we have

$$\ln(2^y) = \ln(x^2y),$$
$$y\ln 2 = 2\ln x+\ln y.$$

Differentiation now gives

$$y'(x)\ln 2 = 2/x+y'(x)/y(x),$$
$$y'(x) = 2y/x(y\ln 2-1). \qquad \square$$

Problem 2.18 Find $y'(x)$ when $y(x) = a^x$ $(a > 0)$.

Solution. Taking logarithms gives $\ln y = x\ln a$, and then differentiation yields $y'(x)/y(x) = \ln a$. Thus $y'(x) = a^x\ln a$.

Note that although y is given explicitly in terms of x it proves advantageous to form an implicit equation by taking logarithms. This is also the case with the following three problems. $\square$

Problem 2.19 Find $y'(x)$ when $y(x) = (x^2)^{\sin x}$.

Solution. Taking logarithms gives $\ln y = \sin x\,.\,2\ln x$. Thus

$$y'(x)/y = \cos x\,.\,2\ln x+2\sin x/x,$$
$$y'(x) = 2(x^2)^{\sin x}(\cos x\ln x+\sin x/x). \qquad \square$$

Problem 2.20 Find $y'(x)$ when

$$y(x) = \frac{\cos^2x\,2^{x^2}(x^3-2x+1)}{\tan x\cosh x\,e^{(x-2)}}.$$

Solution. Taking logarithms gives

$$\ln y = 2\ln\cos x+x^2\ln 2+\ln(x^3-2x+1)-\ln\tan x-\ln\cosh x-(x-2).$$

This enables us to differentiate each term separately, and use the rule of linearity. Thus we have

$$\frac{y'(x)}{y(x)} = \frac{2(-\sin x)}{\cos x}+2x\ln 2+\frac{3x^2-2}{x^3-2x+1}-\frac{\sec^2x}{\tan x}-\frac{\sinh x}{\cosh x}-1.$$

$$y'(x) = \frac{\cos^2x\,2^{x^2}(x^3-2x+1)}{\tan x\cosh x\,.\,e^{(x-2)}}\left[-2\tan x+2x\ln 2+\frac{3x^2-2}{x^3-2x+1}\right.$$
$$\left.-\frac{1}{\cos x\sin x}-\tanh x-1\right] \qquad \square$$

The reader should check that this method gives the same result as that obtained using the product and quotient rules without taking logarithms. It should be noted that one can only take logarithms of positive numbers, within the real number system, and that several of the terms in the formula for $y(x)$ will be negative for particular values of x. The effect of this on the method should be carefully considered. Values of x for which denominators are zero should also be investigated.

Problem 2.21 Find $y'(x)$ when $y = \ln t$ and $x = e^t$.

Solution. In this example y is specified as a function of x by means of the parameter t. Applying the chain rule gives

$$y'(t) = y'(x)\,.\,x'(t),$$

and so $y'(x) = y'(t)/x'(t)$ for values of t for which $x'(t) \neq 0$. In this case $y'(t) = 1/t$ and $x'(t) = e^t$, so that

$$y'(x) = 1/e^t\,t = 1/x\ln x.$$

The parameter t could have been eliminated in this particular problem to obtain $y = \ln\ln x$, and then $y'(x)$ could have been calculated directly. In cases where t cannot be eliminated so easily, $y'(x)$ may have to be given in terms of t or in terms of both x and y. $\square$

Problem 2.22 Find $y'(x)$ when $y = e^t\cos t$ and $x = e^t\sin t$.

Solution. Using the product rule gives

$$y'(t) = -e^t\sin t+e^t\cos t = -x+y,$$
$$x'(t) = e^t\cos t+e^t\sin t = y+x.$$

We thus have

$$y'(x) = y'(t)/x'(t) = (y-x)/(y+x),$$

except for values of t for which $y(t)+x(t) = 0$, i.e. for $t = (n-\frac{1}{4})\pi$, $n = 0, \pm1, \pm2, \ldots$. $\square$

2.7 Higher Derivatives In this chapter we have so far been concerned

with differentiating a function $f(x)$ to obtain the derivative $f'(x)$. This derivative may itself be differentiable for some values of x. and so it may be possible to find a derivative of $f'(x)$. Where possible we can continue in this way and define the nth derivative inductively. The notation is developed in the course of the problems of this section.

Problem 2.23 Find the third derivative of $f(x) = \ln(2x+3)$.

Solution Using the chain rule we have

$$f'(x) = 2/(2x+3),$$
$$f''(x) = 2.(-1).2/(2x+3)^2 = -4/(2x+3)^2,$$
$$f^{(3)}(x) = -4.(-2).2/(2x+3)^3 = 16/(2x+3)^3.$$

Note that each derivative is defined for all values of x except $x = -\frac{3}{2}$, whereas $f(x)$ is defined only for $x > -\frac{3}{2}$. Readers should consider carefully the reasons for this. □

Problem 2.24 Find the nth derivative of $f(x) = \sin x$.

Solution. The first four derivatives of $f(x)$ are given by

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f^{(3)}(x) = -\cos x, \quad f^{(4)}(x) = \sin x.$$

After this stage it would appear that the pattern should repeat itself, and so for any non-negative integer n,

$$f^{(4n)}(x) = \sin x, \qquad f^{(4n+2)}(x) = -\sin x,$$
$$f^{(4n+1)}(x) = \cos x, \qquad f^{(4n+3)}(x) = -\cos x,$$

or more concisely

$$f^{(2n)}(x) = (-1)^n \sin x, \quad f^{(2n+1)}(x) = (-1)^n \cos x.$$

These formulae, which have been guessed on the basis of the pattern of the first four derivatives, can be proved by the method of induction. The details are omitted. □

Problem 2.25 Find the nth derivative of $f(x) = 1/\sqrt{(1-x)}$.

Solution. The first three derivatives, found by successive differentiation are

$$f'(x) = \frac{1}{2}(1-x)^{-\frac{3}{2}}, \quad f''(x) = \frac{1}{2}\cdot\frac{3}{2}(1-x)^{-\frac{5}{2}}, \quad f^{(3)}(x) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-x)^{-\frac{7}{2}},$$

and so one is led to conjecture the general formula

$$f^{(n)}(x) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\ldots\cdot\frac{2n-1}{2}(1-x)^{-\frac{1}{2}(2n+1)} = \frac{(2n)!}{2^{2n}n!}(1-x)^{-(n+\frac{1}{2})}$$

which may be proved by the method of induction. □

Problem 2.26 Find the fourth derivative of $f(x) = x^2e^x$.

Solution. Successive differentiation, using the product rule, gives

$$f'(x) = 2xe^x + x^2e^x,$$
$$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = 2e^x + 4xe^x + x^2e^x,$$
$$f^{(3)}(x) = 2e^x + 4e^x + 4xe^x + 2xe^x + x^2e^x = 6e^x + 6xe^x + x^2e^x,$$
$$f^{(4)}(x) = 6e^x + 6e^x + 6xe^x + 2xe^x + x^2e^x = 12e^x + 8xe^x + x^2e^x. \quad \square$$

It will be seen that with a relatively simple function such as x^2e^x the calculations involve the application of the product rule a number of times. The following theorem, known as Leibniz' Theorem, is of use in this context.

Theorem 2.3 Suppose $h(x) = f(x)g(x)$, where both $f(x)$ and $g(x)$ can be differentiated n times. Then

$$h^{(n)}(x) = f^{(n)}(x)\,g(x) + \binom{n}{1} f^{(n-1)}(x)\,g'(x) + \binom{n}{2} f^{(n-2)}(x)\,g''(x) + \ldots$$
$$+ \binom{n}{r} f^{(n-r)}(x)\,g^{(r)}(x) + \ldots + f(x)\,g^{(n)}(x),$$

where $\binom{n}{r}$ denotes the binomial coefficient $n!/(n-r)!r!$.

Problem 2.27 Find the nth derivative of $h(x) = (x^2-3x+1)/\sqrt{(1-x)}$.

Solution. One of the factors in $h(x)$ is a polynomial, and under these circumstances the application of Leibniz' theorem is especially convenient, since if the polynomial has degree p then the $(p+1)$th and higher derivatives are all zero.

In this example we let $f(x) = 1/\sqrt{(1-x)}$ and $g(x) = x^2-3x+1$. Then $g'(x) = 2x-3$, $g''(x) = 2$, $g^{(3)}(x) = 0$ and so by Th. 2.3,

$$h^{(n)}(x) = f^{(n)}(x)\,g(x) + n\,f^{(n-1)}(x)\,g'(x) + \tfrac{1}{2}n(n-1)\,f^{(n-2)}(x)\,g''(x)$$
$$= \frac{(2n)!}{2^{2n}n!}(1-x)^{-(n+\frac{1}{2})}(x^2-3x+1)$$
$$+ \frac{n(2n-2)!}{2^{2n-2}(n-1)!}(1-x)^{-(n-\frac{1}{2})}(2x-3)$$
$$+ \frac{n(n-1)}{2}\frac{(2n-4)!}{2^{2n-4}(n-2)!}(1-x)^{-(n-3/2)}2,$$
(from Problem 2.25)
$$= \frac{(2n)!}{2^{2n}n!}(1-x)^{-(n+\frac{1}{2})}\frac{3x^2+(2n-9)x-(4n^2-6n-3)}{(2n-1)(2n-3)}. \quad \square$$

***Problem* 2.28** The function $f(x)$ is defined by $f(x) = e^{-1/x^2}(x \neq 0)$, $f(0) = 0$. Prove that $f^{(n)}(0) = 0$ for all n.

Solution. For $x \neq 0$ the rules of differentiation can be applied to show that $f'(x) = 2x^{-3}e^{-1/x^2}$, and it can be proved by the method of induction that for all n,

$$f^{(n)}(x) = P(1/x)e^{-1/x^2} \qquad (x \neq 0),$$

where P is a polynomial. For $x = 0$ we have

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = 0,$$

using Th. 1.2 and Problem 1.17. Assuming that $f^{(n)}(0) = 0$ we then have

$$f^{(n+1)}(0) = \lim_{h\to 0} \frac{f^{(n)}(h)-f^{(n)}(0)}{h} = \lim_{h\to 0} \frac{P(1/h)e^{-1/h^2}}{h} = 0$$

by Th. 1.2 and an extension of Problem 1.17. The result follows for all positive integers n by the principle of mathematical induction. □

EXERCISES

1. Find the derivatives of

   (i) $f(x) = e^{\cos x}$, (ii) $f(x) = \sec x$, (iii) $f(x) = \ln\cosh x$.

2. Investigate differentiability of (i) $f(x) = |x|$, (ii) $f(x) = x^{\frac{1}{3}}$.

3. Find the derivative of $f(x) = \cosh^{-1}x$.

4. Find the derivative of $f(x) = x^x$.

5. Find $y'(x)$ when y satisfies the equation $x^y = y^x$.

6. Find the nth derivative of $f(x) = 1/\sqrt{(ax+b)}$ and prove your result by the method of induction.

7. Find the nth derivative of $f(x) = x^4 \ln x$.

Chapter 3
Applications of the Derivative

3.1 Increasing and Decreasing Functions

Theorem 3.1 (i) If, for all x satisfying $a < x < b$, $f'(x) \geqslant 0$, then $a < x < y < b$ implies $f(x) \leqslant f(y)$, i.e. the function is increasing.

(ii) If, for all x satisfying $a < x < b$, $f'(x) > 0$, then $a < x < y < b$ implies $f(x) < f(y)$, i.e. the function is strictly increasing.

There are corresponding versions of these two results for decreasing functions. There are also variants in which one or more endpoints of the interval between a and b are considered, and a variant of (ii) in which $f'(x) = 0$ at certain points only. These will be used as the occasion arises.

***Problem* 3.1** Find intervals in which the function $f(x) = x^3 - 3x$ is respectively increasing and decreasing.

Solution. For all values of x we have $f'(x) = 3x^2 - 3$. Hence

$$f'(x) > 0 \quad \text{if} \quad 3x^2 - 3 > 0, \quad \text{i.e. } x^2 > 1,$$
$$f'(x) < 0 \quad \text{if} \quad 3x^2 - 3 < 0, \quad \text{i.e. } x^2 < 1.$$

Thus $f(x)$ is increasing when $x < -1$ and when $x > 1$, and decreasing when $-1 < x < 1$. Furthermore the function is continuous at $x = -1$ and $x = 1$, and this is sufficient to ensure that for $x < 1$, $f(x) < f(-1)$, for $-1 < x < 1$, $f(-1) > f(x) > f(1)$, and that for $x > 1$, $f(x) > f(1)$. The graph of $f(x)$ is sketched in Fig. 3.1. □

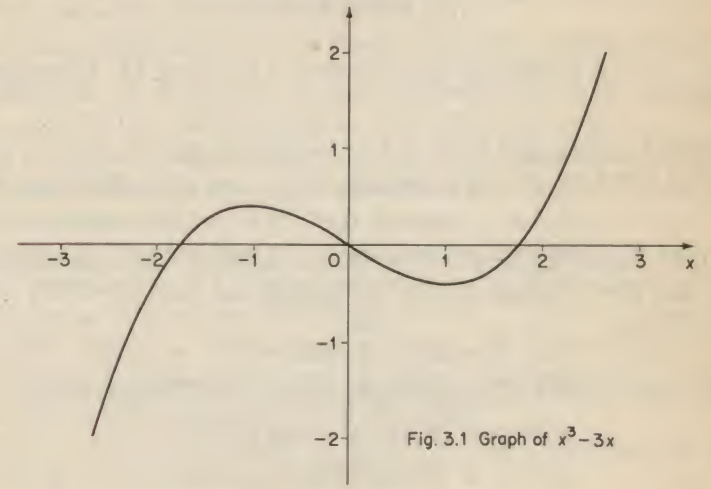

Fig. 3.1 Graph of $x^3 - 3x$

Problem 3.2 Determine those intervals in which the function $f(x) = x^2e^{-x}$ is increasing and those in which it is decreasing.

Solution. For all values of x we have

$$f'(x) = 2xe^{-x} - x^2e^{-x} = e^{-x}x(2-x).$$

Because $e^{-x} > 0$ for all x, it follows that $f'(x) > 0$ if $x(2-x) > 0$, i.e. $0 < x < 2$, and that $f'(x) < 0$ if $x(2-x) < 0$, i.e. if $x < 0$ or $x > 2$. Thus $f(x)$ is increasing in the interval $0 < x < 2$ and decreasing otherwise. Similar considerations to those of Problem 3.1 apply at $x = 0$ and $x = 2$. The graph of $f(x)$ is sketched in Fig. 3.2. (The behaviour of $f(x)$ as $x \to \infty$

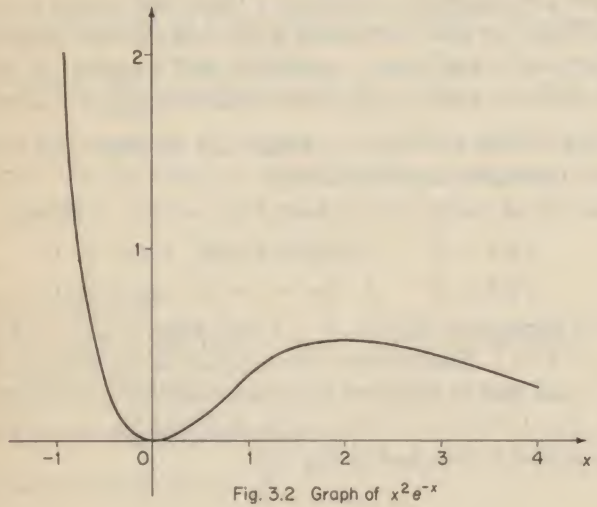

Fig. 3.2 Graph of x^2e^{-x}

and as $x \to -\infty$ is deduced from Problem 1.17, using Th. 1.2 as appropriate.) □

Problem 3.3 Show that for $0 < x < \frac{1}{2}\pi$, $2x/\pi < \sin x < x$.

Solution. (i) Consider the function $f(x) = x - \sin x$. We then have

$$f'(x) = 1 - \cos x > 0, \qquad 0 < x < \tfrac{1}{2}\pi.$$

Thus $f(x)$ is increasing for $0 < x < \frac{1}{2}\pi$. Furthermore $f(x)$ is continuous at $x = 0$ and so a variant of Th. 3.1 establishes that for $0 < x < \frac{1}{2}\pi$, $f(x) > f(0)$, i.e.

$$x - \sin x > 0 - \sin 0 = 0.$$

(ii) For $g(x) = \sin x - (2x/\pi)$ we have $g'(x) = \cos x - (2/\pi)$. Hence

$$g'(x) > 0 \quad \text{if } 0 < x < \cos^{-1}(2/\pi).$$

$$g'(x) < 0 \quad \text{if } \cos^{-1}(2/\pi) < x < \tfrac{1}{2}\pi.$$

Thus $g(x)$ is increasing for $0 < x < \cos^{-1}(2/\pi)$ and decreasing for $\cos^{-1}(2/\pi) < x < \frac{1}{2}\pi$. Continuity of $g(x)$ then implies that

$$g(x) > g(0) = 0 \quad \text{for } 0 < x \leqslant \cos^{-1}(2/\pi),$$

$$g(x) > g(\tfrac{1}{2}\pi) = 0 \quad \text{for } \cos^{-1}(2/\pi) < x < \tfrac{1}{2}\pi.$$

Hence $g(x) > 0$ for $0 < x < \frac{1}{2}\pi$.

Combining the results of (i) and (ii) gives the desired inequalities. □

In Th. 3.1, and in the problems of this section, the sign of the derivative *in an interval* has been investigated. It is important to realize that if $f'(x) > 0$ for a particular value $x = a$, this is not sufficient to guarantee the existence of an interval containing a in which $f(x)$ is increasing. This can be demonstrated by the behaviour, in the region of $x = 0$, of the function defined by

$$f(0) = 0, \qquad f(x) = \tfrac{1}{2}x + x^2\sin(1/x) \quad (x \neq 0).$$

3.2 Maxima and Minima

Definitions (i) A point x at which $f'(x) = 0$ is called a critical point of the function $f(x)$.

(ii) The function $f(x)$ is said to have a local maximum at $x = a$ if there exists a value of d with the property that $f(x) \leqslant f(a)$ for all x satisfying $a - d < x < a + d$.

(iii) The function $f(x)$ is said to have a strict local maximum at $x = a$ if $f(x)$ has a local maximum at $x = a$ and if for $x \neq a$ the inequality in (ii) can be replaced by the strict inequality $f(x) < f(a)$.

(iv) The function $f(x)$ is said to have an absolute maximum at $x = a$ if for all x for which $f(x)$ is defined, $f(x) \leqslant f(a)$.

Corresponding definitions for minima and strict minima can be given.

Theorem 3.2 (i) If $f(x)$ has a local maximum or a local minimum at $x = a$ and if $f(x)$ is differentiable at $x = a$ then $f'(a) = 0$.

(ii) If $f'(a) = 0$, and if $f^{(n)}(a) = 0$ for $n < k$, where k is even, and if $f^{(k)}(a) \neq 0$, then $f(x)$ has a local maximum at $x = a$ if $f^{(k)}(a) < 0$ and a local minimum at $x = a$ if $f^{(k)}(a) > 0$.

Notes: (i) The function $f(x)$ considered in Problem 2.28 has a strict local minimum at $x = 0$. However $f^{(n)}(0) = 0$ for all values of n. Thus there can be no converse of Th. 3.2(ii).

(ii) The behaviour of the function $f(x) = x^3$ in the region of $x = 0$ reveals that in general there may be neither a maximum nor a minimum if the first non-zero derivative is of odd order.

(iii) A critical point which is neither a local maximum nor a local minimum is called a point of inflexion.

Problem 3.4 Find the critical points of $f(x) = x^2e^{-x}$ and classify them.

Solution. The function has been considered in Problem 3.2, and the working there establishes that $f(x)$ has a strict local maximum at $x = 2$, and a strict local minimum, which is in fact a strict absolute minimum, at $x = 0$. Also $f(x)$ has no absolute maximum. We can relate the behaviour at $x = 0$ and $x = 2$ to Th. 3.2 by noting that, since

$$f'(x) = 2xe^{-x} - x^2e^{-x},$$

then $$f'(0) = f'(2) = 0.$$

Also $$f''(x) = e^{-x}(x^2 - 4x + 2)$$

so that $$f''(0) = 2 > 0, \quad f''(2) = -2e^{-2} < 0.$$ □

Problem 3.5 Find the critical points of the function $f(x)$ defined by

$$f(x) = \frac{2x^2 - 3x + 2}{x - 1}$$

and classify them.

Solution. The first thing to notice is that $f(x)$ is not defined for $x = 1$. For all other values of x we apply the quotient rule to obtain

$$f'(x) = \frac{(x-1)(4x-3) - (2x^2 - 3x + 2).1}{(x-1)^2} = \frac{2x^2 - 4x + 1}{(x-1)^2}.$$

Thus $f'(x) = 0$ if and only if $2x^2 - 4x + 1 = 0$, i.e. if and only if $x = 1 + \frac{1}{2}\sqrt{2}$ or $x = 1 - \frac{1}{2}\sqrt{2}$.

Now for $x \neq 1$ we have

$$f''(x) = \frac{(x-1)^2(4x-4) - (2x^2 - 4x + 1)(2x-2)}{(x-1)^4}.$$

Since we shall substitute the critical values of x, and since $(2x^2 - 4x + 1)$ is zero when x takes either of the critical values, there is no point in simplifying the numerator.

$$f''(1 + \tfrac{1}{2}\sqrt{2}) = \frac{(\frac{1}{2}\sqrt{2})^2(2\sqrt{2})}{(\frac{1}{2}\sqrt{2})^4} > 0,$$

$$f''(1 - \tfrac{1}{2}\sqrt{2}) = \frac{(-\frac{1}{2}\sqrt{2})^2(-2\sqrt{2})}{(-\frac{1}{2}\sqrt{2})^4} < 0.$$

Thus by Th. 3.2(ii) with $k = 2$, $f(x)$ has a local minimum at $x = 1 + \frac{1}{2}\sqrt{2}$ and a local maximum at $x = 1 - \frac{1}{2}\sqrt{2}$. □

Problem 3.6 Find the greatest and least values of the function

$$f(x) = \tfrac{1}{2}x + \cos x$$

in the interval from $x = 0$ to $x = 2\pi$.

Solution. Differentiating gives $f'(x) = \frac{1}{2} - \sin x$. Thus $f'(x) = 0$ when $\sin x = \frac{1}{2}$, i.e. $f'(x) = 0$ when $x = \pi/6$ and $x = 5\pi/6$. Also $f''(x) = -\cos x$, so that $f''(\pi/6) = -\frac{1}{2}\sqrt{3} < 0$ and $f''(5\pi/6) = \frac{1}{2}\sqrt{3} > 0$. Thus $f(x)$ has a local maximum at $x = \pi/6$ and a local minimum at $x = 5\pi/6$. The corresponding values of $f(x)$ are

$$f(\pi/6) = \pi/12 + \tfrac{1}{2}\sqrt{3} \simeq 1{\cdot}13, \qquad f(5\pi/6) = 5\pi/12 - \tfrac{1}{2}\sqrt{3} \simeq 0{\cdot}44.$$

Evaluating $f(x)$ at the endpoints of the interval gives

$$f(0) = 1, \qquad f(2\pi) = \pi + 1 \simeq 4{\cdot}14.$$

The function is differentiable throughout the interval and so the greatest value, $\pi + 1$, is attained at the endpoint $x = 2\pi$, while the least value, $5\pi/12 - \frac{1}{2}\sqrt{3}$, is attained at the local minimum $x = 5\pi/6$. The graph of $f(x)$ is sketched in Fig. 3.3. □

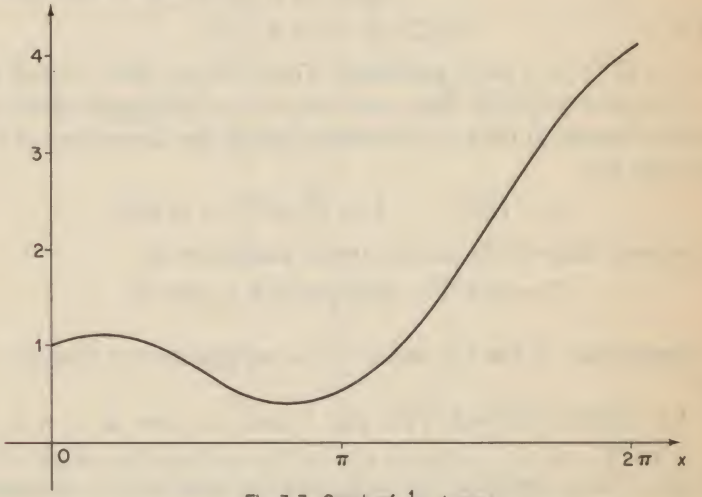

Fig. 3.3 Graph of $\frac{1}{2}x + \cos x$

Problem 3.7 A rectangular box without a top is to be made with a square base and a fixed volume V. Find the dimensions of the box so that the least amount of material is used in its manufacture.

Solution. In a problem such as this we first have to decide what to take as the variables, and what sets of values the variables can take. We then have to formulate equations connecting the variables and find a result from calculations using these equations.

Let the variables, which can take only positive values, be a, the length of a side of the square base of the box, and h, the vertical height. The volume V of the box is then given by

$$V = a^2h.$$

The amount of material required can be measured by the external surface area S of the box, ignoring joints for the sake of simplicity. Since it is to have no top, this will be given by

$$S = a^2+4ah.$$

We now have two equations and we can eliminate one of the variables. Since $h = V/a^2$, we have

$$S = a^2+4aV/a^2 = a^2+4V/a,$$

and must find the least value of the function $S(a)$. Just one critical point occurs, when

$$S'(a) = 2a-4V/a^2 = 0,$$

i.e. when $a^3 = 2V$, or $a = (2V)^{\frac{1}{3}}$. Furthermore

$$S''(a) = 2+8V/a^3,$$

and so $$S''((2V)^{\frac{1}{3}}) = 2+4 > 0.$$

Thus $a = (2V)^{\frac{1}{3}}$ is a local minimum. There are no other critical points, $S(a) \to \infty$ as $a \to 0$, and $S(a) \to \infty$ as $a \to \infty$. The least value of S is therefore attained at the local minimum and so the dimensions of the box in this case are

$$a = (2V)^{\frac{1}{3}}, \qquad h = V/(4V^2)^{\frac{1}{3}} = (V/4)^{\frac{1}{3}}.$$

The minimal value of the surface area is then given by

$$S = (4V^2)^{\frac{1}{3}} + 4(2V)^{\frac{1}{3}}(V/4)^{\frac{1}{3}} = 3(4V^2)^{\frac{1}{3}}.$$ □

3.3 Asymptotes A line l is said to be an asymptote to a function $f(x)$ if either

(i) the distance between $f(x)$ and l tends to zero as $x \to \infty$ or as $x \to -\infty$, or

(ii) the distance between $f(x)$ and l tends to zero as $x \to a$, where a is a number with the property that $f(x) \to \pm\infty$ as $x \to a$ (or as $x \to a$ from above or below).

The meaning of these cases will become clear through the problems below. The way in which the geometrical definition of an asymptote is formulated algebraically will also be explained by means of these problems.

Problem 3.8 Show that the function

$$f(x) = \frac{4x^3 - 3x^2 + 2}{2x^3 - 4x^2 + x - 1}$$

has an asymptote as $x \to \infty$.

Solution. From Problem 1.8 we have $f(x) \to 2$ as $x \to \infty$. Thus the line

$y = 2$ is an asymptote to $f(x)$ as $x \to \infty$. Such an asymptote is called a horizontal asymptote because of its graphical position. □

Problem 3.9 Show that $f(x) = 1/(x-2)$ has an asymptote as $x \to 2$.

Solution. The graph of $f(x)$ is sketched in Fig. 3.4. It will be seen from the graph, and also from the formula, that when x is slightly greater than

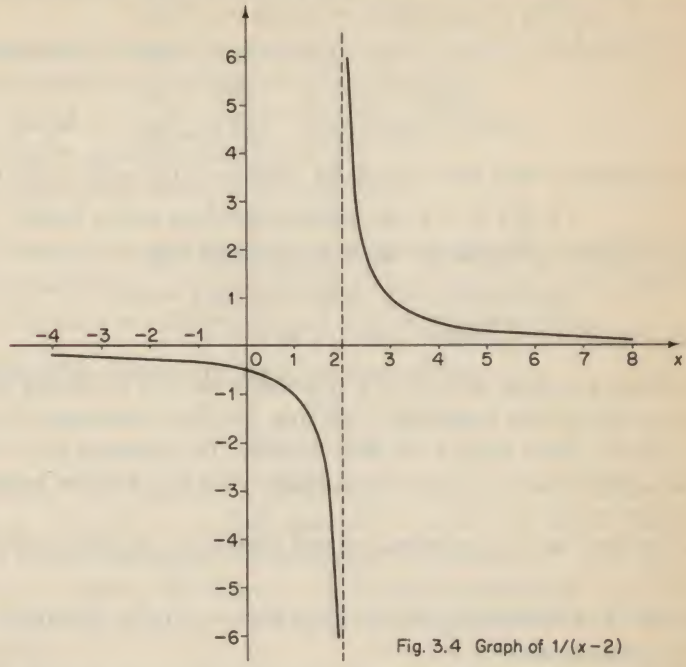

Fig. 3.4 Graph of $1/(x-2)$

2, $f(x)$ is large and positive, and when x is slightly less than 2, $f(x)$ is large and negative. Under these circumstances we say that $f(x) \to \infty$ as $x \to 2$ from above (denoted by $x \to 2+$) and that $f(x) \to -\infty$ as $x \to 2$ from below (denoted by $x \to 2-$). In this case the asymptotic line $x = 2$ is said to be a vertical asymptote.

Note that in this problem the line $y = 0$ is a horizontal asymptote of $f(x)$ as $x \to \infty$ and as $x \to -\infty$. □

Problem 3.10 Show that the line $g(x) = 2x-1$ is an asymptote as $x \to \infty$ and as $x \to -\infty$ of the function $f(x) = (2x^2-3x+2)/(x-1)$.

Solution.

$$f(x) - g(x) = \frac{2x^2 - 3x + 2}{x - 1} - (2x - 1) = \frac{1}{x - 1} \to 0 \quad \text{as } x \to \pm\infty.$$

An asymptote of this kind is called an oblique asymptote. □

Problem 3.11 Show that if $f(x)$ is asymptotic to the line $g(x) = ax + b$ as $x \to \infty$, then $f(x)/x \to a$ as $x \to \infty$. Is the converse true?

Solution. Since

$$f(x) - (ax + b) \to 0 \quad \text{as } x \to \infty,$$

then $$[f(x) - (ax + b)]/x \to 0 \quad \text{as } x \to \infty.$$

But $$[f(x) - (ax + b)]/x = f(x)/x - a + b/x,$$

and $$b/x \to 0 \quad \text{as } x \to \infty.$$

Thus $$f(x)/x - a \to 0 \quad \text{as } x \to \infty,$$

i.e. $$f(x)/x \to a \quad \text{as } x \to \infty.$$

The converse is not true, for if we consider $f(x) = 2x + \sin x$ then

$$f(x)/x = 2 + \sin x/x \to 2 \quad \text{as } x \to \infty.$$

However, there is no value of b with the property that

$$(2x + \sin x) - (2x + b) \to 0 \quad \text{as } x \to \infty,$$

since $\sin x$ oscillates between 1 and -1 as $x \to \infty$. □

This problem provides the key to a systematic method of finding asymptotes, even though the converse is not true. We first investigate $f(x)/x$ as $x \to \infty$. If this has a limit a we then consider the function $f(x) - ax$. If this has a limit b as $x \to \infty$, we can deduce that $f(x)$ has the asymptote $ax + b$ as $x \to \infty$.

A connection between asymptotes and derivatives is given in Problem 4.3.

Problem 3.12 Find whether the function $f(x) = \sqrt{[(x^3 + 3\ln x)/(4x + 1)]}$ has an asymptote as $x \to \infty$.

Solution. Applying the method explained at the end of Problem 3.11 we have

$$\frac{f(x)}{x} = \frac{1}{x}\sqrt{\frac{x^3 + 3\ln x}{4x + 1}} = \sqrt{\frac{x^3 + 3\ln x}{4x^3 + x^2}} = \sqrt{\frac{1 + 3\ln x/x^3}{4 + 1/x}} \to \frac{1}{2} \text{ as } x \to \infty.$$

Now

$$f(x) - \tfrac{1}{2}x = \sqrt{\left[\frac{x^3 + 3\ln x}{4x + 1}\right]} - \tfrac{1}{2}x$$

$$= \frac{(x^3 + 3\ln x)/(4x + 1) - \frac{1}{4}x^2}{\sqrt{[(x^3 + 3\ln x)/(4x + 1)]} + \frac{1}{2}x}$$

$$= \frac{4x^3 + 12\ln x - 4x^3 - x^2}{4(4x + 1)\{\sqrt{[(x^3 + 3\ln x)/(4x + 1)]} + \frac{1}{2}x\}}$$

$$= \frac{12\ln x/x^2 - 1}{4(4 + 1/x)\{\sqrt{[(1 + 3\ln x/x^3)/(4 + 1/x)]} + \frac{1}{2}\}}$$

$$\to \frac{-1}{4 \times 4(\sqrt{\frac{1}{4}} + \frac{1}{2})} = -\frac{1}{16} \quad \text{as } x \to \infty.$$

Thus $f(x)$ has the asymptote $\frac{1}{2}x - \frac{1}{16}$ as $x \to \infty$. □

3.4 Sketching Graphs In drawing a sketch graph of a function $f(x)$ we are concerned to exhibit the following features:

(i) the set of values of x for which $f(x)$ is defined,
(ii) the set of values taken by $f(x)$,
(iii) points of intersection with the axes,
(iv) critical points and their nature,
(v) asymptotes, and the position of the graph with respect to these.

More or less detail concerning these and other features, such as periodicity, will be included as required for any particular application.

Problem 3.13 Sketch the graph of the function

$$f(x) = \frac{2x^2 - 3x + 2}{x - 1}.$$

Solution. The function is defined for all values of x except $x = 1$. When $x = 0$, $f(x) = -2$, but since $2x^2 - 3x + 2 > 0$ for all values of x, $f(x) = 0$ has no solutions. We have $f(x) \to \infty$ as $x \to 1+$ and $f(x) \to -\infty$ as $x \to 1-$, and so the line $x = 1$ is a vertical asymptote. Also $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, so that $f(x)$ has no horizontal asymptote.

From Problem 3.10, the line $2x - 1$ is an oblique asymptote as $x \to \infty$ and as $x \to -\infty$. Furthermore the calculations in that problem show that $f(x) = 2x - 1 + 1/(x - 1)$. Thus if $x > 1$ the graph lies above the asymptote, whilst if $x < 1$ the graph lies below the asymptote.

From Problem 3.5 the only critical points of $f(x)$ are a local maximum at $x = 1 - \frac{1}{2}\sqrt{2}$, and a local minimum at $x = 1 + \frac{1}{2}\sqrt{2}$. Also in that problem we showed that

$$f'(x) = \frac{2x^2 - 4x + 1}{(x - 1)^2} \quad \text{for } x \neq 1.$$

Thus $f'(x) < 0$ if $1 - \frac{1}{2}\sqrt{2} < x < 1 + \frac{1}{2}\sqrt{2}$, except for $x = 1$, and $f'(x) > 0$ otherwise. This information tells us those intervals in which the function is decreasing and increasing respectively. At the critical points we have

$$f(1 + \tfrac{1}{2}\sqrt{2}) = 1 + 2\sqrt{2} \simeq 3{\cdot}8, \qquad f(1 - \tfrac{1}{2}\sqrt{2}) = 1 - 2\sqrt{2} \simeq -1{\cdot}8.$$

The function is continuous for all values of $x \neq 1$, and so we now have enough information to sketch the graph of Fig. 3.5. □

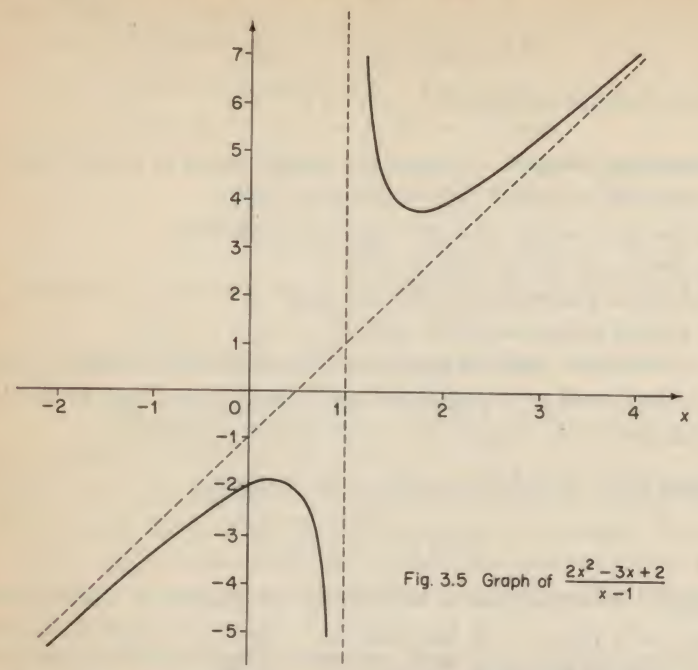

Fig. 3.5 Graph of $\frac{2x^2-3x+2}{x-1}$

Problem 3.14 Sketch the graph of the function $f(x) = x + \ln|x|/x$.

Solution. The function is defined for all values of x except $x = 0$. It will be seen that if we replace x by $-x$ in the formula for $f(x)$, the sign is reversed, i.e. $f(-x) = -f(x)$ for all x for which $f(x)$ is defined. A function with the property is called an *odd* function. (An *even* function is one for which $f(-x) = f(x)$, e.g. $\cos x$.)

Now as $x \to 0+$, $f(x) \to -\infty$, since $\ln|x| \to -\infty$ as $x \to 0$. Also $f(x) \to \infty$ as $x \to 0-$. Thus the line $x = 0$ is a vertical asymptote. Since $f(x) \to \pm\infty$ as $x \to \pm\infty$ there is no horizontal asymptote.

Since $f(1) = 1$ and $f(x) \to -\infty$ as $x \to 0+$, and since $f(x)$ is continuous for $x > 0$, there will be a value of x between 0 and 1 for which $f(x) = 0$. From the formula this value of x is given by the equation $\ln x = -x^2$. We cannot express the solution of this equation explicitly by means of rational numbers, square roots, etc, but we can find rational approximations to the solution from tables or by some other means. How accurate an approximation we require would depend upon the use to which the solution was to be put. For the purposes of sketching a graph a fairly

rough approximation will be sufficient. It can be verified that $x = 0{\cdot}65$ provides such an approximation.

Differentiating $f(x)$ gives, for $x \neq 0$,

$$f'(x) = 1 + \frac{1}{x^2} - \frac{\ln|x|}{x^2} = 1 + \frac{1 - \ln|x|}{x^2}.$$

This is always positive, since $1 + (1 - \ln|x|)/x^2 > 0$ if and only if $\ln|x| < 1 + x^2$, i.e. if and only if $|x| < \exp(1 + x^2)$, which is true since

$$\exp(1 + x^2) = 1 + (1 + x^2) + \ldots > 1 + x^2 > |x|.$$

Hence $f(x)$ is an increasing function of x in the region $x > 0$ and in the region $x < 0$.

To find oblique asymptotes we adopt the procedure of Problem 3.11 which gives

$$f(x)/x = 1 + \ln|x|/x^2 \to 1 \text{ as } x \to \infty.$$

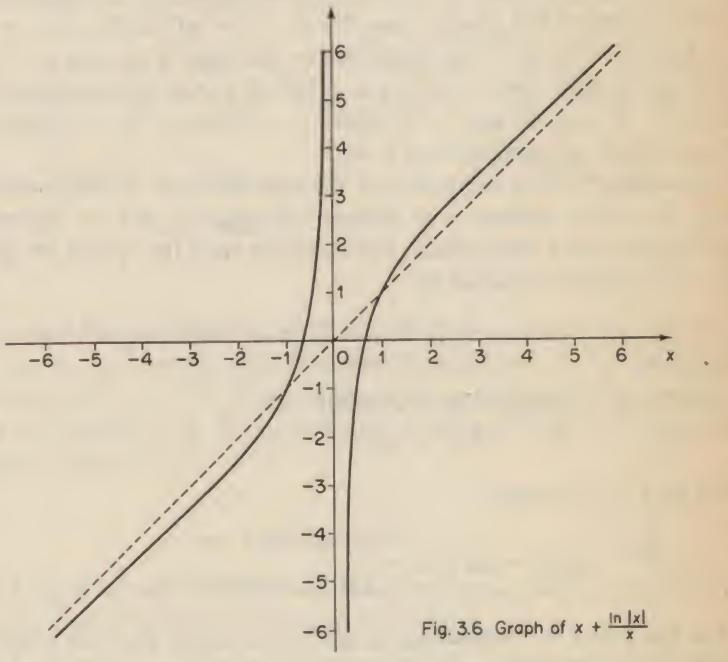

Fig. 3.6 Graph of $x + \frac{\ln|x|}{x}$

Also $$f(x) - x = \ln|x|/x \to 0$$

as $x \to \pm\infty$. Thus the line $g(x) = x$ is an asymptote of $f(x)$ as $x \to \infty$ and as $x \to -\infty$. Since $f(x) = x + \ln|x|/x$ for $x > 0$ we have

$$f(x) < x \text{ when } \ln|x| < 0,$$

and $f(x) > x$ when $\ln|x| > 0$.

Also $f(1) = 1$, and so the graph intersects the oblique asymptote at $x = 1$, lies above it for $x > 1$ and below it for $0 < x < 1$. Since $f(x)$ is an odd function, corresponding results are true for negative values of x. We may now sketch the graph as in Fig. 3.6. □

Problem 3.15 Sketch the curve defined parametrically by the equations

$$x = 1 + t^2, \quad y = \sin t.$$

Solution. As was stated in § 2.6, equations in which x and y are connected implicitly or by means of a parameter do not necessarily define y as a function of x in a unique way. This will be reflected in the graph, where there may be more than one value of y corresponding to a given value of x. This occurs in the present case, for $x(-t) = x(t)$ and $y(-t) = -y(t)$. Since $x(t) \geqslant 1$ for all t, the graph lies to the right of the line $x = 1$. Also, since $y(t) = \sin t$, then $-1 \leqslant y \leqslant 1$ for all t. Also $y(t)$ does not have a limit as $t \to \pm\infty$, i.e. as $x \to \infty$, (since $x \to \infty$ as $t \to \pm\infty$ and *vice versa*). There is thus no asymptote as $x \to \infty$.

Knowledge of the behaviour of the sine function would enable us to draw the curve without any further calculations, but to illustrate the methods we shall give details. Intersections with the x-axis are given by $y(t) = 0$, which is satisfied by

$$t = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and so by $\quad x = 1 + n^2\pi^2 \quad (n = 0, 1, 2, \ldots)$.

To find $y'(x)$ we proceed as in Problem 2.21:

$$y'(t) = \cos t, \qquad x'(t) = 2t.$$

Thus for $t \neq 0$ we have

$$y'(x) = \cos t/2t \to \begin{cases} \infty & \text{as } t \to 0+ \\ -\infty & \text{as } t \to 0-. \end{cases}$$

Since the curve is continuous for all t, this means that the graph has a vertical tangent when $t = 0$, i.e. when $x = 1$.

Now $y'(x) = 0$ when $\cos t = 0$, i.e. $t = (n + \frac{1}{2})\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Further, for $t \neq 0$ we have, using the quotient rule and the chain rule,

$$y''(x) = \frac{2t(-\sin t) - 2\cos t}{4t^2}\,\frac{1}{x'(t)} = -\frac{t\sin t + \cos t}{4t^3}.$$

For $t = (n + \frac{1}{2})\pi$ we have

$$y''(x) = -\frac{(n + \frac{1}{2})\pi(-1)^n}{4[(n+\frac{1}{2})\pi]^3} = -\frac{(-1)^n}{4[(n+\frac{1}{2})\pi]^2}.$$

Thus $y''(x) < 0$ if n is even, and $y''(x) > 0$ if n is odd, so that $y(x)$ has local maxima for

$$t = \ldots, -3\pi/2, 4\pi/2, 5\pi/2, 9\pi/2, \ldots$$

and local minima for

$$t = \ldots, 3\pi/2, -\pi/2, -5/2, -9\pi/2, \ldots.$$

We now have sufficient details to be able to sketch the curve of Fig. 3.7.

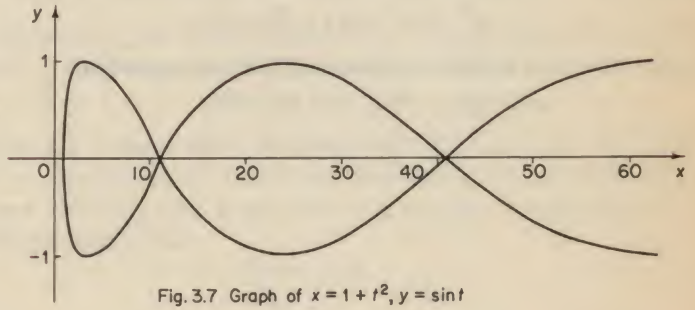

Fig. 3.7 Graph of $x = 1 + t^2$, $y = \sin t$

□

It has only been possible to give a small sample of the types of curve which may be encountered, but many of the important points needed to sketch graphs are covered. More information on curves and their properties is to be found in E. H. Lockwood, *A Book of Curves*, Cambridge University Press, 1961.

EXERCISES

1. Find the intervals in which the function $f(x) = \cos 2x - 2\sin 3x$ is increasing, and intervals in which it is decreasing.

2. Find the critical points of the function $f(x) = (x - 1)^3(x + 2)^4$ and classify them.

3. Find the greatest and least values of the function $f(x) = \ln x + 2/x$ in the interval $1 \leqslant x \leqslant 3$.

4. A kite in the shape of an isosceles triangle is to be fixed by its vertices

to three rods which are joined at a point on the axis of symmetry of the kite. Find a position for this point if the sum of the lengths of the rods is to be minimal.

5. Find asymptotes for the function $f(x) = x \ln(e^a + x^{-b})$, where $a > 0$ and $b > 0$.

6. Find asymptotes for the curve defined by $y = 1/t^2$, $x = t^2/(1 + e^{-t})$.

7. Sketch the graph of the function

$$f(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}.$$

8. Sketch the curve defined by the implicit equation

$$y^3 - 8x^3 + 2y - 3x = 0.$$

9. Sketch the curve defined parametrically by the equations

$$x = t/(1 + t^2),\ y = t^2/(1 + t^2).$$

Chapter 4

Taylor's Theorem

4.1 Mean Value Theorems Notation: We shall call the set of values of x satisfying $a \leqslant x \leqslant b$ the closed interval $[a, b]$. We shall call the set of values of x satisfying $a < x < b$ the open interval (a, b).

Theorem 4.1 Rolle's theorem Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Suppose also that $f(a) = f(b)$. Then there is a number c satisfying $a < c < b$ for which $f'(c) = 0$.

By saying that $f(x)$ is continuous on the closed interval $[a, b]$ we mean that $f(x)$ is continuous for all values of x satisfying $a < x < b$, and that $f(x) \to f(a)$ as $x \to a+$, and also that $f(x) \to f(b)$ as $x \to b-$.

Theorem 4.2 Mean value theorem (M.V.T.) Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then there is a number c satisfying $a < c < b$ with the property that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Problem 4.1 Demonstrate the equivalence of Rolle's theorem and the M.V.T.

Solution. We first show that the M.V.T. implies Rolle's theorem. This follows immediately because the hypotheses of the two theorems are the same, with the additional assumption in Rolle's theorem that $f(a) = f(b)$. Thus, under the assumptions of Rolle's theorem, the M.V.T. implies that there is a number c in (a, b) satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

To show that Rolle's theorem implies the M.V.T. we shall discuss the geometrical significance of each. Fig. 4.1(i) illustrates Rolle's theorem, which may therefore be geometrically interpreted as saying that under the appropriate assumptions, there is a point on the curve where the tangent is parallel to the x-axis. Fig. 4.1(ii) illustrates the M.V.T. The quotient $[f(b) - f(a)]/(b - a)$ is a measure of the slope of the chord, and so the theorem may be geometrically interpreted as saying that there is a point of the curve at which the tangent is parallel to the chord.

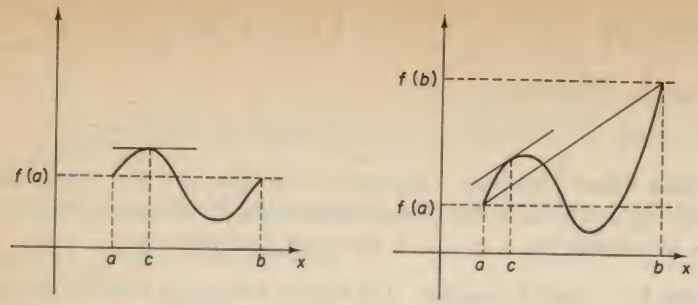

Fig. 4.1

To deduce this from Rolle's theorem we therefore make a transformation which converts the chord into a horizontal line segment. We write

$$g(x) = f(x) - (x-a)\frac{f(b)-f(a)}{b-a}.$$

The function $g(x)$ is continuous on the closed interval $[a,b]$ and differentiable in the open interval (a,b). Furthermore

$$g(b) = f(b) - (b-a)\frac{f(b)-f(a)}{b-a} = f(a),$$

$$g(a) = f(a) - (a-a)\frac{f(b)-f(a)}{b-a} = f(a).$$

Thus $g(b) = g(a)$ and so the function $g(x)$ satisfies the conditions of Rolle's theorem. Hence there is a number c in (a,b) satisfying $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

and so

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0,$$

which is the conclusion of the M.V.T. □

There now follow a number of applications of Rolle's theorem and of the M.V.T.

Problem 4.2 Use the M.V.T. to show that a derived function cannot have a jump discontinuity.

Solution. We assume that $f(x)$ is differentiable in an interval (p,q) and consider a number a in (p,q). To say that $f'(x)$ should have a jump discontinuity at a means that both the limits $\lim_{x\to a+} f'(x)$, $\lim_{x\to a-} f'(x)$ should exist and be different. However, if these limits exist, then for $x > a$, using

the M.V.T. applied to the interval $[a,x]$, there is a number c in (a,x) satisfying $f'(c) = [f(x)-f(a)]/(x-a)$. Now c depends upon x. and since $a < c(x) < x$, $c(x) \to a+$ as $x \to a+$. Thus we have, using a modification of Th. 1.2,

$$\lim_{x\to a+} f'(c(x)) = \lim_{x\to a+} f'(x).$$

However, by definition of $f'(a)$

$$\lim_{x\to a+} f'(c(x)) = \lim_{x\to a+} \frac{f(x)-f(a)}{x-a} = f'(a).$$

A similar argument shows that

$$\lim_{x\to a-} f'(c(x)) = f'(a).$$

Hence if $\lim_{x\to a+} f'(x)$ and $\lim_{x\to a-} f'(x)$ exist, they are both equal to $f'(a)$, and so cannot be different. □

It is tempting to think that this implies that all derived functions are continuous. However, we have used the implication that the existence of $\lim_{x\to a+} f'(x)$ implies the existence of $\lim_{x\to a+} f'(c(x))$, and the converse of this implication is not true in general. An investigation of the behaviour near $x = 0$ of the function $f(x)$ defined by

$$f(x) = x^2 \sin\frac{1}{x} \quad (x \neq 0), \qquad f(0) = 0$$

will show that derived functions can have discontinuities.

Problem 4.3 The function $f(x)$ is assumed to have the following properties: (i) $f(x)$ is differentiable for all x, (ii) $f(x)$ is asymptotic to $ax+b$ as $x \to \infty$, (iii) $\lim_{x\to\infty} f'(x)$ exists. Show that $\lim_{x\to\infty} f'(x) = a$.

Solution. Let k be a fixed real number. We apply the M.V.T. to the interval $[k,x]$ where $x > k$. Thus there is a number $c(x)$ in (k,x) satisfying

$$\frac{f(x)-f(k)}{x-k} = f'(c(x)).$$

Since $f'(x)$ has a limit as $x \to \infty$ it follows that $f'(c(x))$ has the same limit as $x \to \infty$. We now compare $f(x)$ with $ax+b$. Since they are asymptotic, $f(x)-(ax+b) \to 0$ as $x \to \infty$. Thus

$$\frac{f(x)-f(k)}{x-k} = \frac{f(x)-(ax+b)}{x-k} + \frac{ax+b}{x-k} - \frac{f(k)}{x-k}$$

$$\to 0 + a + 0 = a \quad \text{as } x \to \infty.$$

Hence $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} f'(c(x)) = \lim_{x\to\infty} \dfrac{f(x)-f(k)}{x-k} = a.$ □

Readers should construct an example of a function $f(x)$ which satisfies assumptions (i) and (ii) but not assumption (iii).

Problem 4.4 Discuss possible numbers and location of solutions for the equation $x^n + px^2 + q = 0 \quad (p \neq 0, q \neq 0)$.

Solution. Problems of this type depend upon Rolle's theorem. If $f'(x) = 0$ has only one solution then there can be at most two solutions of $f(x) = 0$. Writing

$$f(x) = x^n + px^2 + q,$$

we then have

$$f'(x) = nx^{n-1} + 2px$$
$$= 0, \quad \text{when } x = 0 \text{ or } x^{n-2} = -2p/n.$$

The following situations can now arise:

(a) n even, $p > 0$; $f'(x) = 0$ has only one solution and so $f(x) = 0$ has at most two solutions.

(b) n even, $p < 0$; $f'(x) = 0$ has three solutions and so $f(x) = 0$ can have at most four solutions.

(c) n odd; $f'(x) = 0$ has two solutions and so $f(x) = 0$ can have at most three solutions.

In case (a), $f(x)$ is an even function and $f(x) \to \infty$ as $x \to \pm\infty$. $f(x) = 0$ then has either no solutions or two solutions, as in Fig. 4.2 (i) and (ii) respectively.

In case (b), $f(x)$ is again even, and $f(x) = 0$ may have 0, 2 or 4 solutions as in Fig. 4.2 (iii), (iv) and (v) respectively.

In case (c), $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$ and so there must be at least one solution of $f(x) = 0$ since $f(x)$ is continuous for all x. There may be only one solution as in Fig. 4.2 (vi) and (vii), or three solutions as in Fig. 4.2 (viii) (corresponding to $p < 0$) or as in Fig. 4.2 (ix) (corresponding to $p > 0$). The positions of the solutions have been located by the principle that a pair of solutions associated with a solution of $f'(x) = 0$ lie either side of the value of x for which $f'(x) = 0$. □

4.2 Indeterminate Forms—l'Hôpital's Rule

Theorem 4.3 Cauchy's M.V.T. The functions $f(x)$ and $g(x)$ are assumed to be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). We assume also that $g(b) \neq g(a)$ and that $f'(x)$ and $g'(x)$ are not both zero for the same value of x. Then there is a number c in (a, b) satisfying

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

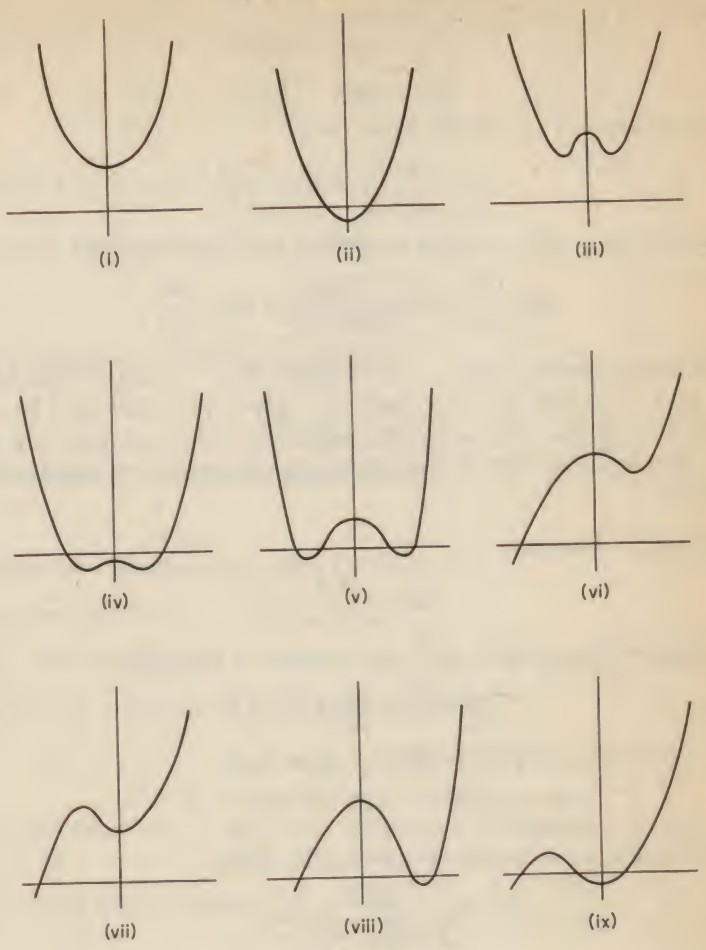

Fig. 4.2

Problem 4.5 Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f'(x)/g'(x)$ exists. Show that

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x).$$

This result is known as l'Hôpital's rule.

Solution. We shall define $f(a) = g(a) = 0$, since nothing about the values of $f(a)$ and $g(a)$ has been assumed in the hypotheses. Since the limit

of $f'(x)/g'(x)$ as $x \to a$ is being considered, this presupposes the existence of $f'(x)$ and $g'(x)$ in some interval containing a. We apply Cauchy's M.V.T. to obtain, for all x in such an interval,

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}.$$

As in Problem 4.2, c depends upon x and

$$\lim_{x\to a} \frac{f'(c(x))}{g'(c(x))} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Moreover $f(a) = g(a) = 0$ and so the last two equations give

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(c(x))}{g'(c(x))} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$ □

This result applies when $x \to \infty$ instead of $x \to a$, and it also applies when $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$. Many problems may be transformed so that the rule is directly applicable, and examples are given below. Readers may care to formulate general procedures based on these problems.

Problem 4.6 Evaluate

$$\lim_{x\to 1} \frac{\ln x}{\sin(x-1)}.$$

Solution. Writing $f(x) = \ln x$, $g(x) = \sin(x-1)$, we have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = 0.$$

Also $f'(x) = 1/x$ and $g'(x) = \cos(x-1)$, so that

$$\lim_{x\to 1} f'(x)/g'(x) = \lim_{x\to 1} 1/x\cos(x-1) = 1.$$

Thus l'Hôpital's rule is directly applicable, giving

$$\lim_{x\to 1} \frac{\ln x}{\sin(x-1)} = 1.$$ □

Problem 4.7 Evaluate

$$\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2}.$$

Solution. Writing $f(x) = \cos^2 x - 1$, $g(x) = x^2$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

Also $f'(x) = -2\cos x \sin x$ and $g'(x) = 2x$ so that

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0.$$

However, $f''(x) = 2\sin^2 x - 2\cos^2 x$ and $g''(x) = 2$. Thus

$$\lim_{x\to 0} f''(x) = -2 \quad \text{and} \quad \lim_{x\to 0} g''(x) = 2.$$

Repeated application of l'Hôpital's rule now gives

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{-2}{2} = -1.$$ □

Problem 4.8 Evaluate $\lim_{x\to 1} x^{2/\sin(x-1)}$.

Solution. Let $h(x) = x^{2/\sin(x-1)}$, then

$$\ln h(x) = \frac{2\ln x}{\sin(x-1)}.$$

As in Problem 4.6, l'Hôpital's rule gives

$$\lim_{x\to 0} \frac{2\ln x}{\sin(x-1)} = 2.$$

Thus $\lim_{x\to 1} \ln h(x) = 2$, so that $\lim_{x\to 1} h(x) = e^2$. □

Problem 4.9 Evaluate $\lim_{x\to \frac{1}{2}\pi} (\tan x - \sec x)$

Solution. We have

$$\lim_{x\to \frac{1}{2}\pi+} \tan x = \lim_{x\to \frac{1}{2}\pi+} \sec x = -\infty, \qquad \lim_{x\to \frac{1}{2}\pi-} \tan x = \lim_{x\to \frac{1}{2}\pi-} \sec x = \infty.$$

However, we transform the problem as follows,

$$\tan x - \sec x = \tan x - \frac{1}{\cos x} = \frac{\cos x \tan x - 1}{\cos x} = \frac{\sin x - 1}{\cos x}$$

Now $\lim_{x\to \frac{1}{2}\pi} (\sin x - 1) = \lim_{x\to \frac{1}{2}\pi} \cos x = 0$ and so l'Hôpital's rule gives

$$\lim_{x\to \frac{1}{2}\pi} (\tan x - \sec x) = \lim_{x\to \frac{1}{2}\pi} \frac{\sin x - 1}{\cos x} = \lim_{x\to \frac{1}{2}\pi} \frac{\cos x}{-\sin x} = 0.$$ □

It is important to verify that the requisite conditions are satisfied before applying l'Hôpital's rule, or errors will result. For example, if $f(x) = x^2 + 2x + 4$ and $g(x) = 3x^3 + 2x$, then l'Hôpital's procedure would yield

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{2x+2}{9x^2+2} = \frac{4}{11},$$

whereas in fact $\lim_{x\to 1} f(x)/g(x) = 7/5$. The error arises because $f(x)$ and $g(x)$ both have non-zero finite limits as $x \to 1$. so that the rule is inapplicable.

4.3 The Taylor Series

Theorem 4.4 Taylor's theorem Let $f(x)$ be a function with the properties that $f'(x), f''(x), \ldots, f^{(n-1)}(x)$ all exist and are continuous on the closed interval $[a, b]$ and that $f^{(n)}(x)$ exists in (a, b). Then

$$f(b) = f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\ldots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+R_n(b)$$

where the remainder term $R_n(b)$ may be given in several forms, including

$$R_n(b) = \frac{f^{(n)}(a+\theta(b-a))}{n!}(b-a)^n \qquad \text{(Lagrange's form)},$$

$$R_n(b) = \frac{f^{(n)}(a+\phi(b-a))}{(n-1)!}(1-\phi)^{n-1}(b-a)^n \qquad \text{(Cauchy's form)},$$

where θ and ϕ are numbers between 0 and 1, so that $a < a+\theta(b-a) < b$.

In many applications we require an expression involving powers of x, and we use the particular case of Taylor's theorem with $a = 0$ and $b = x$. We then obtain

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+R_n(x),$$

where $$R_n(x) = \frac{f^{(n)}(\theta x)}{n!}x^n \qquad \text{(Lagrange)}$$

or $$R_n(x) = \frac{f^{(n)}(\phi x)}{(n-1)!}(1-\phi)^{n-1}x^n \qquad \text{(Cauchy)}.$$

This result is sometimes known as Maclaurin's theorem. The infinite series

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

is called the Taylor expansion of $f(x)$ about $x = 0$.

If in the formulation of Taylor's theorem we abbreviate the notation to $f(x) = S_n(x)+R_n(x)$, then we must clearly have

$$\lim_{n\to\infty}[S_n(x)+R_n(x)] = f(x).$$

However, the separate limits $\lim_{n\to\infty} S_n(x)$ and $\lim_{n\to\infty} R_n(x)$ need not exist, although if one limit exists then so must the other. For example if $\lim_{n\to\infty} R_n(x)$ exists, then writing $S_n(x) = f(x)-R_n(x)$ we see that

$$\lim_{n\to\infty} S_n(x) = f(x)-\lim_{n\to\infty} R_n(x).$$

Thus either both limits exist, or neither exists. When both limits exist we

have

$$f(x) = \lim_{n\to\infty} S_n(x)+\lim_{n\to\infty} R_n(x),$$

and so $f(x)$ will be equal to $\lim_{n\to\infty} S_n(x)$, the sum of the infinite series, if and and only if $\lim_{n\to\infty} R_n(x) = 0$.

We have not discussed the properties of infinite series in this book, neither have we defined the sum of an infinite series. For readers who have not met these notions, it should suffice at this stage to remark that the definition of $\lim_{n\to\infty} R_n$ is very similar to a definition such as $\lim_{x\to\infty} f(x)$, given in §1.1, the main difference being that the variable n is restricted to positive integer values.

Problem 4.10 Find the Taylor expansion about $x = 0$ of $f(x) = \ln(1+x)$ and determine the values of x for which $R_n(x) \to 0$.

Solution. We first note that $f(x)$ is defined only for $x > -1$. For these values of x we then have

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f^{(3)}(x) = \frac{2}{(1+x)^3}, \quad f^{(4)}(x) = \frac{-2.3}{(1+x)^4}.$$

The pattern established leads us to conjecture that

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n},$$

and this formula can be proved by the method of induction. We therefore have $f^{(n)}(0) = (-1)^{n+1}(n-1)!$ so that the Taylor series for $\ln(1+x)$ is

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n+1}\frac{x^n}{n}+\ldots.$$

From the theory of infinite series we use the result that this series has a sum only for $-1 < x \leqslant 1$, and so we need only be concerned with these values of x. The remainder term is given in Cauchy's form by

$$R_n(x) = \frac{(-1)^{n+1}(n-1)!}{(1+\phi x)^n}\frac{(1-\phi)^{n-1}}{(n-1)!}x^n = \frac{(-1)^{n+1}(1-\phi)^{n-1}x^n}{(1+\phi x)^n},$$

where $0 < \phi < 1$.

$$R_n(1) = \frac{(-1)^{n+1}(1-\phi)^{n-1}}{(1+\phi)^n} = \frac{(-1)^{n+1}}{(1-\phi)}\left(\frac{1-\phi}{1+\phi}\right)^n \to 0 \quad \text{as } n \to \infty,$$

since $0 < (1-\phi)/(1+\phi) < 1$. For $0 \leqslant x < 1$ we have

$$|R_n(x)| = \frac{(1-\phi)^{n-1}x^n}{(1+\phi x)^n} \leqslant x^n \to 0 \qquad \text{as } n \to \infty.$$

For $-1 < x < 0$ we have

$$|R_n(x)| = \frac{(1-\phi)^{n-1}|x|^n}{(1-\phi|x|)^n} = \frac{1}{1-\phi}\left(\frac{|x|-\phi|x|}{1-\phi|x|}\right)^n \to 0 \quad \text{as } n \to \infty.$$

since $0 < (|x|-\phi|x|)/(1-\phi|x|) < 1$. Hence $R_n(x) \to 0$ as $n \to \infty$ for all values of x satisfying $-1 < x \leqslant 1$. □

Problem 4.11 Find the Taylor expansion about $x = 0$ of the function $f(x)$ defined by

$$f(x) = e^{-1/x^2} \quad (x \neq 0), \qquad f(0) = 0,$$

and determine those values of x for which $R_n(x) \to 0$.

Solution. From Problem 2.28 $f^{(n)}(0) = 0$ for all values of n, and so the Taylor series $f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\ldots$ is identically zero. Furthermore, since $f(x) = S_n(x)+R_n(x)$ we must have $S_n(x) = 0$ and $R_n(x) = f(x)$ for all values of n and all values of x. Thus if $x \neq 0$, $R_n(x) \not\to 0$. □

This problem is important as it shows that even when a Taylor series possesses an infinite sum, that sum need not be equal to the value of the function.

4.4 Errors and Approximations An important application of Taylor's theorem is that it provides a polynomial approximation for a function $f(x)$. In using such an approximation we are concerned with possible errors, and in the problems in this section we estimate the error from the magnitude of the remainder term.

Problem 4.12 Find a polynomial approximation for $\ln(1+x)$ for $0 \leqslant x \leqslant 1/10$ with an error of at most 10^{-6} in magnitude.

Solution. Using the Taylor series for $\ln(1+x)$ derived in Problem 4.10, the remainder $R_n(x)$ is given in Cauchy's form by

$$R_n(x) = \frac{(-1)^{n+1}(1-\phi)^{n-1}x^n}{(1+\phi x)^n},$$

and so for $x \geqslant 0$ we have

$$|R_n(x)| = \frac{(1-\phi)^{n-1}x^n}{(1+\phi x)^n} \leqslant x^n.$$

Thus for $0 \leqslant x \leqslant 1/10$ we have $|R_n(x)| \leqslant 1/10^n$. In order to achieve $|R_n(x)| < 10^{-6}$ we must therefore take $n = 6$, and the requisite polynomial approximation is

$$P(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

We can now evaluate approximately $\ln(1+x)$ for any value of x in $[0, 1/10]$. For example, taking $x = 1/10$ we have

$$\ln 1{\cdot}1 = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} - \frac{1}{40\,000} + \frac{1}{500\,000}$$
$$= 0{\cdot}1 - 0{\cdot}005 + 0{\cdot}000\,\dot{3} - 0{\cdot}000\,025 + 0{\cdot}000\,002 = 0{\cdot}095\,310\,\dot{3}.$$

The error $|R_6(1/10)|$ is at most 10^{-6}, giving a possible error of ± 1 in the sixth decimal place. We can therefore say that $\ln 1{\cdot}1 = 0{\cdot}09531$ correct to 5 decimal places. □

Problem 4.13 Evaluate $\sqrt[3]{8{\cdot}03}$ correct to six decimal places.

Solution. We shall first find the Taylor series for $f(x) = \sqrt[3]{x}$, taking $a = 8$.

$$f(x) = f(8)+f'(8)(x-8)+\frac{f''(8)}{2!}(x-8)^2+\ldots+\frac{f^{(n-1)}(8)}{(n-1)!}(x-8)^{n-1}+R_n(x),$$

where, using Lagrange's form of the remainder,

$$R_n(x) = \frac{f^{(n)}(8+\theta(x-8))}{n!}(x-8)^n.$$

Now since $f(x) = x^{\frac{1}{3}}$,

$$f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}}, \quad f''(x) = -\tfrac{1}{3}.\tfrac{2}{3}.x^{-\frac{5}{3}}, \quad f^{(3)}(x) = \tfrac{1}{3}.\tfrac{2}{3}.\tfrac{5}{3}.x^{-\frac{8}{3}}.$$

The pattern established leads to the conjecture that

$$f^{(n)}(x) = (-1)^{n+1}\frac{1.2.5.\ldots.(3n-4)}{3^n}x^{-n+\frac{1}{3}},$$

which may be proved by the method of induction. Thus we obtain

$$|R_n(x)| = \frac{1.2.5.\ldots.(3n-4)}{3^n}\frac{(8+0{\cdot}03\theta)^{-n+\frac{1}{3}}}{n!}(0{\cdot}03)^n$$
$$< \frac{1.2.5.\ldots.(3n-4)}{3^n.n!}8^{-n+\frac{1}{3}}\left(\frac{3}{100}\right)^n$$
$$= \frac{1.2.5.\ldots.(3n-4)}{n!}\frac{2}{8^n}\frac{1}{100^n}$$
$$< \frac{1.3.6.9.\ldots.(3n-3)}{n!}\frac{2}{8^n}\frac{1}{100^n}$$
$$= \frac{3^{n-1}.2}{n.800^n} < 10^{-7} \quad \text{provided } n \geqslant 3.$$

Thus the first three terms of the expansion will provide the required degree

of accuracy, and so

$$\sqrt[3]{8{\cdot}03} \simeq 8^{\frac{1}{3}} + \tfrac{1}{3}.8^{-\frac{2}{3}}(0{\cdot}03) - \tfrac{1}{3}.\tfrac{2}{3}.\frac{8^{-\frac{5}{3}}}{2!}(0{\cdot}03)^2$$

$$= 2+(0{\cdot}01)/4-(0{\cdot}01)^2/32$$

$$= 2 + 0{\cdot}0025 - 0{\cdot}000\,003\,125 = 2{\cdot}002\,496\,875.$$

Since $|R_3(8{\cdot}03)| < 10^{-7}$ there is a maximum possible error of $\pm$ 1 in the seventh decimal place and so $\sqrt[3]{8{\cdot}08} = 2{\cdot}002\,497$ correct to six decimal places.

It should be noticed that in estimating the error we have increased parts of the remainder term in order to simplify the calculations, without increasing the number of terms of the Taylor series which have to be evaluated. □

EXERCISES

1. Verify that the following functions satisfy the conditions of Rolle's theorem in the stated intervals, and find the values of c in those intervals for which $f'(c) = 0$.

   (i) $f(x) = \cos x$ in $[-2\pi, 2\pi]$;
   (ii) $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$.

2. Investigate possible numbers and location of solutions of the equation $x^n + px + q = 0$.

3. Evaluate the following limits:

   (i) $\lim_{x\to 0} \dfrac{\ln \sin 5x}{\ln \sin 2x}$ (ii) $\lim_{x\to 0} \dfrac{e^{\sinh x} - e^x}{\sinh x - x}$ (iii) $\lim_{x\to \frac{1}{2}\pi} (\sin^2 x)^{1/(x-\frac{1}{2}\pi)^2}$.

4. Find the Taylor expansion about $x = 0$ of $\sinh x$ and evaluate $\sinh(0{\cdot}05)$ correct to five places of decimals.

Chapter 5

Integration

5.1 Definitions Let $f(x)$ be a function defined on the interval $[a, b]$. If there is a function $F(x)$ with the property that, for all values of x in the interval $[a, b]$, $F(x)$ is differentiable and $F'(x) = f(x)$, then $F(x)$ is called an indefinite integral (or primitive) of $f(x)$ over $[a, b]$. If $G(x)$ is another indefinite integral of $f(x)$ over $[a, b]$, then it can be shown that there is a number c such that for all values of x in $[a, b]$, $F(x) - G(x) = c$. We shall write

$$\int f(x)\,dx = F(x)$$

where the equality symbol here indicates that $F(x)$ is one of the indefinite integrals of $f(x)$ over $[a, b]$. When, as is often the case, $f(x)$ is defined for all real values of x and $F'(x) = f(x)$ for all real x, we drop the reference to $[a, b]$ and say that $F(x)$ is an indefinite integral of $f(x)$. The integral is easily shown to be linear in the sense that if $F(x) = \int f(x)\,dx$ and $G(x) = \int g(x)\,dx$ then for any real numbers h and k,

$$h\,F(x) + k\,G(x) = \int (hf(x) + kg(x))\,dx.$$

The definite integral of $f(x)$ over an interval $[p, q]$ is defined as $F(q) - F(p)$. This number is independent of which indefinite integral $F(x)$ of $f(x)$ is used. We write

$$F(q) - F(p) = \int_p^q f(x)\,dx.$$

5.2 Simple Extensions In this section we give a number of problems in which the function whose integral is to be found can be reduced by algebraic manipulation so that one of the standard formulae from the table of indefinite integrals can be applied.

The table, which is given on the next page, has been built up by using a knowledge of differentiation to find a function $F(x)$ such that $F'(x) = f(x)$. In the third column allowable intervals are indicated. Thus the third entry denotes that $\ln|x|$ is an indefinite integral of $1/x$ over any interval consisting entirely of positive numbers, or entirely of negative numbers, but not over any interval including zero.

Table of indefinite integrals

$f(x)$	$F(x)$	allowable intervals
c = constant	cx	all real x
$x^a \quad (a \neq -1)$	$x^{a+1}/a+1$	all real x
$1/x$	$\ln\|x\|$	$x \neq 0$
$\sin x$	$-\cos x$	all real x
$\cos x$	$\sin x$	all real x
$\sec^2 x$	$\tan x$	$x \neq (n+\frac{1}{2})\pi \ (n = 0, \pm 1, \pm 2, \ldots)$
e^x	e^x	all real x
$\cosh x$	$\sinh x$	all real x
$\sinh x$	$\cosh x$	all real x
$\operatorname{sech}^2 x$	$\tanh x$	all real x
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$	all real x
$\dfrac{1}{x^2-a^2}$	$\dfrac{-1}{a}\coth^{-1}\dfrac{x}{a}$	$\|x\| > \|a\|$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\dfrac{x}{a}$	$\|x\| < \|a\|$
$\dfrac{1}{\sqrt{(x^2+a^2)}}$	$\sinh^{-1}\dfrac{x}{a}$	all real x
$\dfrac{1}{\sqrt{(x^2-a^2)}}$	$\cosh^{-1}\dfrac{x}{a}$	$\|x\| > \|a\|$
$\dfrac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\dfrac{x}{a}$	$\|x\| < \|a\|$

We first give three general results.

Problem 5.1 Show that if $F(x) = \int f(x)\,dx$ then

$$\frac{1}{a}F(ax+b) = \int f(ax+b)\,dx.$$

Solution. As with all equations involving indefinite integrals, the relation $F(x) = \int f(x)\,dx$ is to be interpreted as meaning that for all values of x in an appropriate interval $F'(x) = f(x)$. Thus using the chain rule, if

$$G(x) = \frac{1}{a}F(ax+b),$$

we have

$$G'(x) = \frac{1}{a}F'(ax+b).\,a = f(ax+b).$$

Hence

$$\frac{1}{a}F(ax+b) = \int f(ax+b)\,dx. \qquad \square$$

Problem 5.2 Show that if $F(x) = \int f(x)\,dx$ then $F(g(x)) = \int f(g(x))\,g'(x)\,dx$.

Solution. Let $G(x) = F(g(x))$. Using the chain rule we obtain

$$G'(x) = F'(g(x))\,g'(x) = f(g(x))\,g'(x),$$

since $F(x) = \int f(x)\,dx$ stands for $F'(x) = f(x)$. Hence

$$F(g(x)) = G(x) = \int f(g(x))\,g'(x)\,dx.$$

The result of Problem 5.1 can be deduced from this result by letting $g(x) = ax+b$ and using linearity. $\square$

Problem 5.3 Show that if $G(x) = \int g(x)\,dx$ then $\ln|G(x)| = \int [g'(x)/g(x)]\,dx$.

Solution. This is a particular case of the result of Problem 5.2 obtained by putting $f(x) = 1/x$; it is a case which is often useful. $\square$

Problem 5.4 Find an indefinite integral of $f(x) = (2x^3 - 4x + 1)/(x-1)$.

Solution. By division we have

$$\frac{2x^3-4x+1}{x-1} = 2x^2 + 2x - 2 - \frac{1}{x-1}.$$

Linearity therefore gives

$$\int \frac{2x^3-4x+1}{x-1}\,dx = 2\int x^2\,dx + 2\int x\,dx - \int 2\,dx - \int \frac{dx}{x-1}$$

$$= \tfrac{2}{3}x^3 + x^2 - 2x - \ln|x-1|, \quad \text{valid for } x \neq 1. \qquad \square$$

Problem 5.5 Find an indefinite integral of $f(x) = \sin^4 x \cos x$.

Solution. This is a direct application of Problem 5.2 with $f(x) = x^4$ and $g(x) = \sin x$. Thus, since $x^5/5 = \int x^4 dx$, we have

$$\frac{1}{5}\sin^5 x = \int \sin^4 x \cos x\,dx. \qquad \square$$

Problem 5.6 Find an indefinite integral of $f(x) = e^{\sqrt{x}}/\sqrt{x}$.

Solution. Apart from a constant factor $f(x)$ is the derived function of $e^{\sqrt{x}}$. In fact, if $g(x) = e^{\sqrt{x}}$, the chain rule gives

$$g'(x) = e^{\sqrt{x}}.\tfrac{1}{2}x^{-\frac{1}{2}} = \tfrac{1}{2}f(x).$$

Hence

$$\int f(x)\,dx = 2g(x) = 2e^{\sqrt{x}}. \qquad \square$$

Problem 5.7 Find an indefinite integral of $f(x) = x\sqrt{(1+x^2)}$.

Solution. This may be done as in Problem 5.6, or regarded as a special case of Problem 5.2, with $f(x) = \sqrt{(1+x)}$ and $g(x) = x^2$. We then have

$$\int x\sqrt{(1+x^2)}\,dx = \tfrac{1}{2}\int 2x\sqrt{(1+x^2)}\,dx = \tfrac{1}{2}\cdot\tfrac{2}{3}(1+x^2)^{\frac{3}{2}} = \tfrac{1}{3}(1+x^2)^{\frac{3}{2}}. \qquad \square$$

Problem 5.8 Find an indefinite integral of $f(x) = \sin mx \cos nx$.

Solution. Many examples of this type rely on the use of trigonometrical identities to reduce the function to a standard form. We use the identity

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

to obtain $\sin mx \cos nx = \tfrac{1}{2}\sin(m+n)x + \tfrac{1}{2}\sin(m-n)x$.

We then have, using linearity and Problem 5.1,

$$\int \sin mx \cos nx\, dx = \tfrac{1}{2}\int \sin(m+n)x\, dx + \tfrac{1}{2}\int \sin(m-n)\, x\, dx$$

$$= \frac{-\cos(m+n)x}{2(m+n)} + \frac{-\cos(m-n)x}{2(m-n)} \qquad \square$$

Problem 5.9 Find an indefinite integral for $f(x) = 1/(a - \cos x)$, where $a > 1$.

Solution. We use the trigonometrical identity $\cos x = 1 - 2\sin^2\tfrac{1}{2}x$. We then have

$$\int \frac{dx}{a-\cos x} = \int \frac{dx}{2\sin^2\tfrac{1}{2}x - 1 + a} = \int \frac{\sec^2\tfrac{1}{2}x\, dx}{2\tan^2\tfrac{1}{2}x + (a-1)\sec^2\tfrac{1}{2}x}$$

$$= \int \frac{\sec^2\tfrac{1}{2}x\, dx}{(a+1)\tan^2\tfrac{1}{2}x + (a-1)} = \frac{1}{a+1}\int \frac{\sec^2\tfrac{1}{2}x\, dx}{\tan^2\tfrac{1}{2}x + (a-1)/(a+1)}$$

$$= \frac{1}{a+1}\sqrt{\left(\frac{a+1}{a-1}\right)} \times 2\tan^{-1}\left(\tan\tfrac{1}{2}x\sqrt{\frac{a+1}{a-1}}\right)$$

$$= \frac{2}{\sqrt{(a^2-1)}}\tan^{-1}\left(\tan\tfrac{1}{2}x\sqrt{\frac{a+1}{a-1}}\right),$$

using Problem 5.2 together with a standard formula. $\square$

Problem 5.10 Find an indefinite integral for $f(x) = (2x+3)/(x^2-3x+2)$.

Solution. Using the algebraic device of partial fractions we have

$$\frac{2x+3}{x^2-3x+2} = \frac{2x+3}{(x-2)(x-1)} = \frac{7}{x-2} - \frac{5}{x-1}.$$

We then have

$$\int \frac{2x+3}{x^2-3x+2}\, dx = \int \frac{7}{x-2}\, dx - \int \frac{5}{x-1}\, dx = 7\ln|x-2| - 5\ln|x-1|,$$

the result being valid over any interval which excludes the numbers $x = 1$ and $x = 2$. $\square$

Problem 5.11 Find an indefinite integral of $f(x) = (2x+3)/(x^2-4x+5)$.

Solution. Unlike Problem 5.10 the denominator does not have real roots. We complete the square to obtain $x^2-4x+5 = (x-2)^2+1$. We then have

$$\int \frac{2x+3}{x^2-4x+5}\, dx = \int \frac{2x-4}{x^2-4x+5}\, dx + \int \frac{7}{(x-2)^2+1}\, dx$$

$$= \ln|x^2-4x+5| + 7\tan^{-1}(x-2),$$

using Problem 5.3 and a standard formula. $\square$

All rational functions may eventually be reduced to polynomials and combinations of functions of the types dealt with in the previous problems.

Problem 5.12 Find an indefinite integral of $f(x) = (x+2)/\sqrt{(x^2-4x+3)}$.

Solution. As in Problem 5.11, we may complete the square in the denominator. We then write

$$\int \frac{x+2}{\sqrt{(x^2-4x+3)}}\, dx = \int \frac{x-2}{\sqrt{(x^2-4x+3)}}\, dx + \int \frac{4}{\sqrt{((x-2)^2-1)}}\, dx$$

$$= \sqrt{(x^2-4x+3)} + 4\cosh^{-1}(x-2),$$

the result being valid over any interval which does not contain points of the interval $[1, 3]$. $\square$

5.3 Changing the Variable This method is a consequence of the chain rule for differentiation and was derived for indefinite integrals in Problem 5.2. What was not stressed in that example, however, was that in changing the variable the intervals over which the integration is valid may be altered. This is particularly important when dealing with definite integrals. A full discussion of the theory is outside the scope of this book; it involves a careful consideration of the role of possible inverses of the function $x = g(t)$ used to change the variable.

Problem 5.13 Find an indefinite integral for $f(x) = \sqrt{x}/(\sqrt{x} - \sqrt[3]{x})$.

Solution. We make an algebraic substitution which transforms the function into a rational function. Let $x(t) = t^6$, then $x'(t) = 6t^5$. The equation $\int f(x)\, dx = \int f(x(t))\, x'(t)\, dt$ applied in the present example gives

$$\int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}}\, dx = \int \frac{t^3 6t^5}{t^3 - t^2}\, dt = \int \frac{6t^6}{t-1}\, dt$$

$$= 6\int \left(t^5 - t^4 + t^3 - t^2 + t - 1 + \frac{1}{t-1}\right) dt$$

$$= 6\frac{t^6}{6} - 6\frac{t^5}{5} + 6\frac{t^4}{4} - 6\frac{t^3}{3} + 6\frac{t^2}{2} - 6t + 6\ln|t-1|$$

$$= x - \tfrac{6}{5}\sqrt[6]{x^5} + \tfrac{3}{2}\sqrt[3]{x^2} - 2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1|.$$

□

Problem 5.14 Find an indefinite integral for $f(x) = (x^2-1)/x\sqrt{(x-2)}$.

Solution. We make a substitution to remove the square root. Let $x(t) = t^2+2$, then $x'(t) = 2t$. We then have

$$\int \frac{x^2-1}{x\sqrt{(x-2)}}\,dx = \int \frac{(t^2+2)^2-1}{(t^2+2)t}\,2t\,dt = 2\int\left(t^2+2-\frac{1}{t^2+2}\right)dt$$

$$= 2\frac{t^3}{3} + 4t - \frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}$$

$$= \tfrac{2}{3}\sqrt{(x-2)^3} + 4\sqrt{(x-2)} - \sqrt{2}\tan^{-1}\sqrt{(\tfrac{1}{2}x-1)}. \quad \square$$

Problem 5.15 Find an indefinite integral for $f(x) = \sqrt{(b^2-x^2)}$.

Solution. This is an example of an integral which can be evaluated by one of the important class of trigonometrical substitutions. Let $x(t) = b\sin t$, then $x'(t) = b\cos t$. We then have

$$\int \sqrt{(b^2-x^2)}\,dx = \int \sqrt{(b^2-b^2\sin^2 t)}\,b\cos t\,dt = \int b^2\cos^2 t\,dt$$

$$= \tfrac{1}{2}b^2\int(\cos 2t + 1)\,dt = \tfrac{1}{2}b^2(\tfrac{1}{2}\sin 2t + t)$$

$$= \tfrac{1}{2}b^2\sin t\cos t + \tfrac{1}{2}b^2 t$$

$$= \tfrac{1}{2}x\sqrt{(b^2-x^2)} + \tfrac{1}{2}b^2\sin^{-1}(x/b). \quad \square$$

Problem 5.16 Find an indefinite integral for $f(x) = 1/(2\sin x + \cos x)$.

Solution. This, and many similar integrals, may be evaluated by the transformation $x(t) = 2\tan^{-1} t$, i.e. $t = \tan\frac{1}{2}x$. We then have

$$x'(t) = \frac{2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2},$$

so that

$$\int \frac{dx}{2\sin x + \cos x} = \int \frac{2\,dt}{2.2t + 1 - t^2} = \int \frac{2\,dt}{1+4t-t^2}$$

$$= \int \frac{-2\,dt}{(t-2-\sqrt{5})(t-2+\sqrt{5})}$$

$$= \int \frac{-1\,dt}{(t-2-\sqrt{5})\sqrt{5}} + \int \frac{dt}{(t-2+\sqrt{5})\sqrt{5}}$$

$$= -(1/\sqrt{5})\ln|(t-2-\sqrt{5})| + (1/\sqrt{5})\ln|(t-2+\sqrt{5})|$$

$$= -(1/\sqrt{5})\ln|(\tan\tfrac{1}{2}x - 2 - \sqrt{5})| + (1/\sqrt{5})\ln|(\tan\tfrac{1}{2}x - 2 + \sqrt{5})|.$$

□

Problem 5.17 Evaluate the integral

$$\int_0^{\pi/4} \frac{dx}{4\cos^2 x + 3\sin^2 x}$$

Solution. We make the substitution $x = \tan^{-1} t$, i.e. $t = \tan x$. In evaluating a definite integral it is not necessary to change back to the original variable as we have done in the previous four examples, instead we alter the interval of integration. The interval $[0, \pi/4]$ is transformed by the substitution $x = \tan^{-1} t$ into the interval $[0, 1]$. The integral thus becomes

$$\int_0^{\pi/4} \frac{\sec^2 x}{4 + 3\tan^2 x}\,dx = \int_0^1 \frac{dt}{4+3t^2} = \frac{1}{3}\int_0^1 \frac{dt}{t^2+4/3}$$

$$= [\tfrac{1}{3}\sqrt{(\tfrac{3}{4})}\tan^{-1}\sqrt{(\tfrac{3}{4})}\,t]_0^1 = (1/\sqrt{12})\tan^{-1}\sqrt{\tfrac{3}{4}}. \quad \square$$

The relationship between the indefinite and definite integrals has been indicated by the use of the notation $[F(x)]_p^q$ to stand for $F(q) - F(p)$. We first find an indefinite integral and then evaluate it at the two endpoints of the interval of integration.

5.4 Integration by Parts In the previous section we considered a method arising from the chain rule. The method of this section is a consequence of the product rule for differentiation. The formula to be applied is

$$\int f(x)\,g(x)\,dx = f(x)\,G(x) - \int f'(x)\,G(x)\,dx,$$

where $G'(x) = g(x)$, i.e. $G(x)$ is an indefinite integral of $g(x)$. The central problem in applying this result to evaluate $\int h(x)\,dx$ is to express $h(x)$ as a product $h(x) = f(x)\,g(x)$ in such a way that the right-hand side of the formula can be evaluated with ease.

Problem 5.18 Find an indefinite integral of the function $h(x) = xe^x$.

Solution. We apply the formula of integration by parts using $f(x) = x$ and $g(x) = e^x$. We then have $G(x) = e^x$ and so

$$\int xe^x\,dx = xe^x - \int 1.e^x\,dx = xe^x - e^x. \quad \square$$

Problem 5.19 Find an indefinite integral for the function $h(x) = x^2\cos x$.

Solution. In this example it proves necessary to integrate by parts twice.

In the first instance we set $f(x) = x^2$ and $g(x) = \cos x$ so that $G(x) = \sin x$. We then have

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx.$$

We now apply the formula to evaluate $\int 2x \sin x \, dx$ and obtain

$$\begin{aligned}\int x^2 \cos x \, dx &= x^2 \sin x - 2x(-\cos x) + \int 2(-\cos x)\, dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x. \end{aligned}$$ □

Problem 5.20 Find an indefinite integral for $h(x) = (\ln^2 x)\sqrt{x}$.

Solution. In this example it is not immediately clear whether it is most advantageous to set $f(x) = \sqrt{x}$ or $f(x) = \ln^2 x$. In a situation such as this one must try both approaches. For this particular example if we try $f(x) = \sqrt{x}$ and $g(x) = \ln^2 x$ it becomes apparent that we cannot find easily an indefinite integral $G(x)$ for $g(x) = \ln^2 x$. We therefore set $f(x) = \ln^2(x)$ and $g(x) = \sqrt{x}$ so that $G(x) = \frac{2}{3}x^{\frac{3}{2}}$. We then have

$$\begin{aligned}\int (\ln^2 x)\sqrt{x}\, dx &= (\ln^2 x)\tfrac{2}{3}x^{\frac{3}{2}} - \int 2(\ln x)(1/x)\tfrac{2}{3}x^{\frac{3}{2}}\, dx \\ &= (\ln^2 x)\tfrac{2}{3}x^{\frac{3}{2}} - (4/3)\int (\ln x)x^{\frac{1}{2}}\, dx.\end{aligned}$$

We now integrate by parts once more to obtain

$$\begin{aligned}\int (\ln^2 x)\sqrt{x}\, dx &= (\ln^2 x)\tfrac{2}{3}x^{\frac{3}{2}} - (4/3)\left[(\ln x)\tfrac{2}{3}x^{\frac{3}{2}} - \int (1/x)\tfrac{2}{3}x^{\frac{3}{2}}\, dx\right] \\ &= \tfrac{2}{3}(\ln^2 x)x^{\frac{3}{2}} - (8/9)(\ln x)x^{\frac{3}{2}} + (16/27)x^{\frac{3}{2}}.\end{aligned}$$ □

Problem 5.21 Find an indefinite integral for $h(x) = \ln x$.

Solution. It is not clear from the nature of the function $h(x)$ that integration by parts is applicable. However, by setting $f(x) = \ln x$ and $g(x) = 1$ we obtain

$$\int \ln x \, dx = x \ln x - \int (1/x)x \, dx = x \ln x - x.$$ □

Problem 5.22 Find an indefinite integral for $h(x) = e^x \cos 2x$.

Solution. We let $f(x) = \cos 2x$ and $g(x) = e^x$. We then obtain, on integrating by parts twice,

$$\begin{aligned}\int \cos 2x.e^x\, dx &= \cos 2x.e^x - \int -2\sin 2x.e^x\, dx = \cos 2x.e^x + 2\int \sin 2x.e^x\, dx \\ &= e^x \cos 2x + 2(\sin 2x.e^x - \int 2\cos 2x.e^x\, dx) \\ &= e^x \cos 2x + 2e^x \sin 2x - 4\int \cos 2x.e^x\, dx.\end{aligned}$$

This seemingly brings us back to the integral we began with. However, we now have a relation which we can rearrange to give

$$5\int e^x \cos 2x\, dx = e^x \cos 2x + 2e^x \sin 2x,$$

i.e.
$$\int e^x \cos 2x\, dx = \tfrac{1}{5}e^x \cos 2x + \tfrac{2}{5}e^x \sin 2x.$$ □

5.5 Reduction Formulae In some of the problems in the previous section it was necessary to integrate by parts twice. The problems in this section give rise to integration by parts n times. In such cases it is often algebraically convenient if the integral $I_n = \int f_n(x)\, dx$ can be expressed in terms of I_{n-1} and possibly in terms also of integrals I_r with $r < n-1$. Such an expression is termed a reduction formula.

Problem 5.23 Find a reduction formula for $I_n = \int (\ln x)^n\, dx$, and use it to evaluate $\int_1^e (\ln x)^4\, dx$.

Solution. We write $f(x) = (\ln x)^n$ and $g(x) = 1$. Integrating by parts then gives

$$I_n = (\ln x)^n x - \int n(\ln x)^{n-1}(1/x)x\, dx = (\ln x)^n x - nI_{n-1}.$$

This gives the required reduction formula. To find the definite integral we apply the formula to give

$$\int_1^e (\ln x)^n\, dx = (\ln e)^n e - (\ln 1)^n 1 - n\int_1^e (\ln x)^{n-1}\, dx.$$

Thus, since $\ln 1 = 0$, we have

$$\begin{aligned}\int_1^e (\ln x)^4\, dx &= e - 4\int_1^e (\ln x)^3\, dx \\ &= e - 4\left[e - 3\int_1^e (\ln x)^2\, dx\right] \\ &= -3e + 12\left[e - 2\int_1^e \ln x\, dx\right] \\ &= 9e - 24[(e \ln e - e) - (1 \ln 1 - 1)] \\ &= 9e - 24, \quad \text{using Problem 5.21.}\end{aligned}$$ □

Problem 5.24 Find reduction formulae for the integrals

$$I_n = \int x^n \sin x\, dx, \qquad J_n = \int x^n \cos x\, dx.$$

Solution. To find an indefinite integral for $x^n \sin x$, let $f(x) = x^n$,

$g(x) = \sin x$, and integrate by parts to obtain

$$I_n = x^n(-\cos x) - \int nx^{n-1}(-\cos x)\,dx = -x^n\cos x + nJ_{n-1}.$$

Similarly

$$J_n = x^n\sin x - nI_{n-1}.$$

To obtain a formula involving only Is we use the second equation to eliminate the Js as follows:

$$\begin{aligned} I_n &= -x^n\cos x + nJ_{n-1} = -x^n\cos x + n(x^{n-1}\sin x - (n-1)I_{n-2}) \\ &= -x^n\cos x + nx^{n-1}\sin x - n(n-1)I_{n-2}. \end{aligned}$$

□

5.6 Differentiation of Integrals There are basically two situations in which integrals are differentiated. The first is where the variable occurs in the function to be integrated, and the second is where the variable occurs as an end-point of the interval of integration of a definite integral. We quote first two theorems which give conditions under which the processes are valid, and then solve one problem containing both types of differentiation.

Theorem 5.1 Let $f(x) = \int_a^x g(t)\,dt$. If the function $g(t)$ is continuous at x_0. i.e. if $\lim_{t\to x_0} g(t) = g(x_0)$, then $f(x)$ is differentiable at x_0 and $f'(x_0) = g(x_0)$.

Every function which we have encountered in the problems in this chapter has been continuous in the intervals of integration, and so this theorem is widely applicable.

Theorem 5.2 Let $f(x, t)$ be continuous as a function of t for $c \leqslant t \leqslant d$, for each x in some interval $[a, b]$. Suppose also that the partial derivative $f_x(x, t)$ is continuous in the region $a < x < b$, $c \leqslant t \leqslant d$. Then if

$$F(x) = \int_c^d f(x, t)\,dt,$$

we have

$$F'(x) = \int_c^d f_x(x, t)\,dt$$

for all x in (a, b).

An acquaintance with the definition of a partial derivative and its first simple manipulations is all that is required for Problem 5.25, which is worked in much more detail than would be needed when familiar with the operations. (Partial derivatives are treated in L. Marder, *Calculus of Several Variables*, in this series.)

Problem 5.25 Let

$$F(x) = \left(\int_0^x \exp(-t^2)\,dt\right)^2, \qquad G(x) = \int_0^1 \frac{\exp[-x^2(t^2+1)]}{t^2+1}\,dt.$$

Evaluate $F'(x)$ and $G'(x)$ and use these to find a relation between $F(x)$ and $G(x)$. Deduce that

$$\lim_{x\to\infty} \int_0^x \exp(-t^2)\,dt = \tfrac{1}{2}\sqrt{\pi}.$$

Solution. Write

$$f(x) = \int_0^x \exp(-t^2)\,dt.$$

Then $F(x) = (f(x))^2$ and, from the chain rule, $F'(x) = 2f(x)f'(x)$. We can evaluate $f'(x)$ using Th. 5.1, since $\exp(-t^2)$ is continuous for all t.

Thus we have $f'(x) = \exp(-x^2)$ and so

$$F'(x) = 2\exp(-x^2)\int_0^x \exp(-t^2)\,dt.$$

For $G'(x)$ we use Th. 5.2. Writing $f(x, t) = \exp[-x^2(t^2+1)]/(t^2+1)$ it follows that

$$f_x(x, t) = \frac{-2x(t^2+1)\exp[-x^2(t^2+1)]}{t^2+1} = -2x\exp[-x^2(t^2+1)].$$

Hence

$$\begin{aligned} G'(x) &= \int_0^1 -2x\,\exp(-x^2t^2)\exp(-x^2)\,dt \\ &= -2\exp(-x^2)\int_0^1 x\exp(-x^2t^2)\,dt. \end{aligned}$$

(We may take $\exp(-x^2)$ outside the integral since we are integrating with respect to t.) We now change the variable, letting $u(t) = xt$, to obtain

$$G'(x) = -2\exp(-x^2)\int_0^x \exp(-u^2)\,du.$$

It follows that $F'(x) + G'(x) = 0$, and so $F(x) + G(x) = \text{constant}$. To find the value of this constant we put $x = 0$. Clearly

$$F(0) = 0, \qquad G(0) = \int_0^1 dt/(1+t^2) = \tan^{-1}1 = \tfrac{1}{4}\pi.$$

Thus for all x, $F(x) + G(x) = \frac{1}{4}\pi$. We shall now prove that $G(x) \to 0$ as $x \to \infty$. For all t in the interval $[0, 1]$ we have

$$0 \leqslant \exp[-x^2(t^2+1)]/(t^2+1) \leqslant \exp(-x^2).$$

Thus $0 \leqslant G(x) \leqslant \exp(-x^2)$ and so $G(x) \to 0$ as $x \to \infty$ since

$$\exp(-x^2) \to 0 \quad \text{as} \quad x \to \infty.$$

Thus

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} (F(x)+G(x)) - \lim_{x\to\infty} G(x) = \tfrac{1}{4}\pi - 0.$$

It follows that

$$\lim_{x\to\infty}\int_0^x \exp(-t^2)\,dt = \sqrt{(\tfrac{1}{4}\pi)} = \tfrac{1}{2}\sqrt{\pi}.$$

Results such as this properly belong to the theory of infinite integrals, which lies outside the scope of this book. □

5.7 The Integral as a Sum In this section we outline an alternative method of defining a definite integral which is important for many geometrical and physical applications. We shall not give any analytical proofs in this section, but refer readers to a book which deals with Riemann integration (e.g. D. A. Quadling, *Mathematical Analysis*, Ch. X, Oxford University Press, 1955).

Let $[a,b]$ be a closed interval and let $f(x)$ be a function defined on $[a,b]$. A *dissection* of $[a,b]$ is a finite set of numbers $\{x_0,x_1,x_2,\ldots,x_n\}$ satisfying $a = x_0 < x_1 < \ldots < x_n = b$. Let t_r be any number satisfying $x_{r-1} \leqslant t_r \leqslant x_r$ $(r = 1,2,\ldots,n)$. The theory of the Riemann integral then says that for an appropriate type of function $f(x)$ the sum

$$\sum_{r=1}^{n} f(t_r)(x_r - x_{r-1})$$

tends to a limit as the length of the largest interval of the dissection tends to zero. This limiting process is more complicated than that introduced in Chapter 1, and its details are omitted here. It can be shown that if the function $f(x)$ can be integrated over $[a,b]$ in the sense of § 5.1 then this limit is equal to the integral obtained by the method of that section. We shall therefore write

$$\sum_{r=1}^{n} f(t_r)(x_r - x_{r-1}) \to \int_a^b f(x)\,dx$$

as $\max(x_r - x_{r-1}) \to 0$.

***Problem* 5.26** Evaluate $\int_0^2 x^3\,dx$ by summation.

Solution. We subdivide the interval $[0,2]$ into n equal parts so that $x_r = 2r/n$ $(r = 0,1,2,\ldots,n)$. We may take t_r to be any point in the interval $[x_{r-1},x_r]$, and so we shall choose $t_r = x_{r-1}$. We then have to evaluate the sum

$$\sum_{r=1}^{n} f(t_r)(x_r - x_{r-1}) = \sum_{r=1}^{n}\left(\frac{2(r-1)}{n}\right)^3\frac{2}{n} = \frac{16}{n^4}\sum_{r=1}^{n}(r-1)^3$$

$$= \frac{16(n-1)^2n^2}{n^4\,4} = 4\left(1-\frac{1}{n}\right)^2 \to 4 \qquad \text{as } n\to\infty.$$

As $n \to \infty$ the lengths of the intervals of the subdivision tend to zero and

so we have $\int_0^2 x^3\,dx = 4$, a result readily confirmed by the usual methods of integration. To illustrate the possibility of choosing alternative values for t_r, we shall carry out the calculations with $t_r = x_r$ instead of $t_r = x_{r-1}$. In that case

$$\sum_{r=1}^{n} f(t_r)(x_r - x_{r-1}) = \sum_{r=1}^{n}\left(\frac{2r}{n}\right)^3\frac{2}{n} = \frac{16}{n^4}\sum_{r=1}^{n} r^3$$

$$= \frac{16}{n^4}\frac{n^2}{4}(n+1)^2 = 4\left(1+\frac{1}{n}\right)^2 \to 4 \qquad \text{as } n\to\infty. \ \square$$

***Problem* 5.27** By dividing the interval $[1,2]$ at the points

$$x_r = 2^{r/n} \quad (r = 0,1,2,\ldots,n)$$

prove that $\int_1^2 dx/x^2 = \frac{1}{2}$. Hence deduce that

$$\lim_{n\to\infty} n\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}\right) = \frac{1}{2}.$$

Solution. We take $t_r = x_{r-1}$ and the sum corresponding to the subdivision is then

$$\sum_{r=1}^{n} f(t_r)(x_r - x_{r-1}) = \sum_{r=1}^{n}\frac{1}{(2^{(r-1)/n})^2}(2^{r/n} - 2^{(r-1)/n})$$

$$= \sum_{r=1}^{n}\frac{2^{1/n}-1}{2^{(r-1)/n}}$$

$$= (2^{1/n}-1)\left[1-\left(\frac{1}{2^{1/n}}\right)^n\right]\Bigg/\left[1-\frac{1}{2^{1/n}}\right]$$

$$= 2^{1/n}(1-\tfrac{1}{2}) \to \tfrac{1}{2} \quad \text{as } n\to\infty.$$

Hence $\int_1^2 dx/x^2 = \frac{1}{2}$. If we now subdivide the interval into equal parts by taking $x_r = 1 + r/n$ $(r = 0,1,2,\ldots,n)$ and $t_r = x_r$, we obtain the sum

$$\sum_{r=1}^{n} f(t_r)(x_r - x_{r-1}) = \sum_{r=1}^{n}\frac{1}{(1+r/n)^2}\frac{1}{n} = n\sum_{r=1}^{n}\frac{1}{(n+r)^2}.$$

The theory tells us that this sum must also tend to $\int_1^2 dx/x^2 = \frac{1}{2}$ as $n\to\infty$, and this evaluates the required limit. □

EXERCISES

1. Find indefinite integrals for the following functions:

(i) $x(x^2+3)^4$ (ii) $\tan x$ (iii) $\dfrac{x^3+2x^2-4}{x^2-5x+6}$ (iv) $\dfrac{3x^2-2x+1}{x^2+x+6}$

2. Find indefinite integrals for the following functions by changing the variable:

(i) $\dfrac{1}{a+b\cosh x}$ $(a > b)$ (ii) $\dfrac{x+3}{x+2\sqrt{(x-1)}}$ (iii) $(a^2-x^2)^{\frac{3}{2}}$

3. Find indefinite integrals for the following functions by the method of integration by parts:

(i) $x\tan^{-1}x$ (ii) $x^2\cosh x$ (iii) $\cos x\cosh x$ (iv) $\sin^{-1}x$

4. Find relations between the integrals $I = \int e^{ax}\sin bx\,dx$ and $J = \int e^{ax}\cos bx\,dx$ by integration of each by parts. Hence find explicit formulae for I and J.

5. Show that if $I_n = \tan^n x\,dx$ then $I_n = \dfrac{\tan^{n-1}x}{n-1} + I_{n-2}$. Use this formula to evaluate $\displaystyle\int_0^{\pi/4} \tan^7 x\,dx$.

6. Find the limit as $n \to \infty$ of

$$a_n = \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

Chapter 6

Geometrical and Physical Applications of Integration

In applying the process of integration to geometrical and physical situations we shall not develop all the theoretical aspects of the work. We shall quote certain results and regard the chapter mainly as an illustration of the techniques introduced in Chapter 5.

6.1 Plane Areas

Problem 6.1 Find the area of the region bounded by the parabolas $f(x) = 2x^2$, $g(x) = 3x^2+5$, and the line $y = 8$.

Solution. Fig. 6.1 is a sketch of the region. Considerations of symmetry

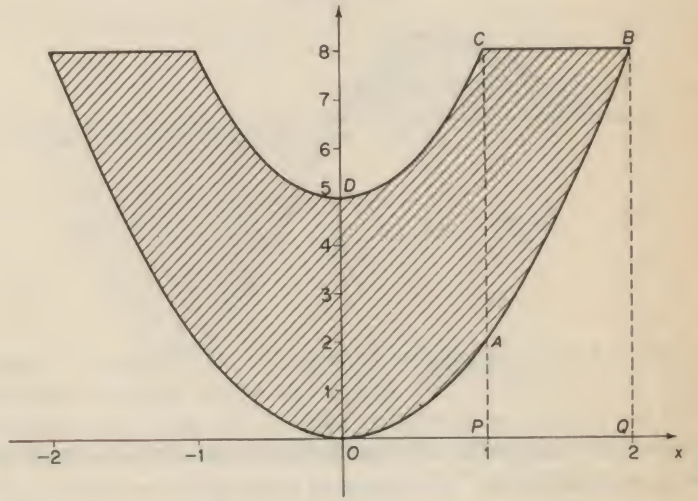

Fig. 6.1

enable the required area A to be built up as follows, denoting by $a(R)$ the area of a region R.

$$\begin{aligned} A &= 2\,a(OABCD) = 2[a(OACD)+a(ABC)] \\ &= 2[a(OPCD)-a(OPA)+a(PQBC)-a(PQBA)] \\ &= 2\left[\int_0^1 g(x)\,dx-\int_0^1 f(x)\,dx+PQ.QB-\int_1^2 f(x)\,dx\right] \end{aligned}$$

$$= 2\left[\int_0^1 (3x^2+5)\,dx + 8 - \int_0^2 2x^2\,dx\right]$$

$$= 2\{[x^3+5x]_0^1 + 8 - [\tfrac{2}{3}x^3]_0^2\} = 2(6+8-5\tfrac{1}{3}) = 17\tfrac{1}{3}. \quad \square$$

The geometrical result utilized in this problem is that the region bounded by the non-negative function $f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$ has an area given by $\int_a^b f(x)\,dx$. If $f(x)$ is negative then the value of the integral will also be negative. In finding areas, therefore, care must be taken over the sign of the function, as in the following problem.

Problem 6.2 Find the area of the region bounded by the curve $f(x) = x\sin x$ and the x-axis between $x = 0$ and $x = 2\pi$.

Solution. The region is sketched in Fig. 6.2. It comprises two parts.

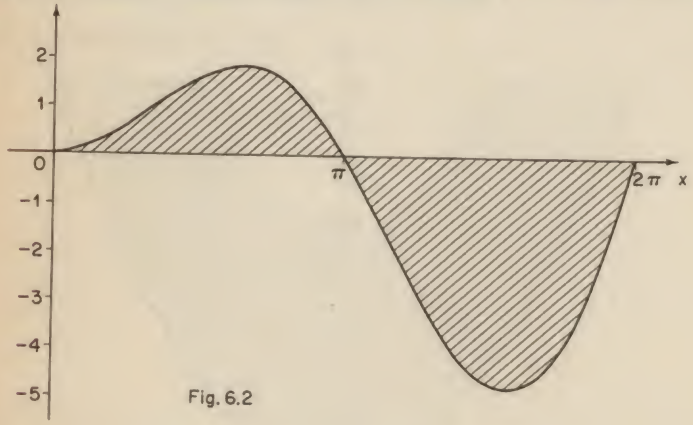

Fig. 6.2

The areas of the two parts will be calculated separately, taking the sign of $f(x)$ into consideration.

$$\int_0^{\pi} x\sin x\,dx = [x(-\cos x)]_0^{\pi} - \int_0^{\pi} (-\cos x)\,dx$$

$$= \pi - [-\sin x]_0^{\pi} = \pi.$$

$$-\int_{\pi}^{2\pi} x\sin x\,dx = -[x(-\cos x)]_{\pi}^{2\pi} + [-\sin x]_{\pi}^{2\pi} = 3\pi.$$

Thus the total area of the region is 4π. Without taking the sign of $f(x)$ into account, the calculation of $\int_0^{2\pi} f(x)\,dx$ would lead to the incorrect result of -2π. $\square$

6.2 Volumes of Revolution The volume of a solid obtained by rotating the plane region bounded by the graph of the non-negative function $f(x)$, the lines $x = a$, $x = b$, and also the x-axis, about the x-axis, is given by the formula

$$V = \pi\int_a^b (f(x))^2\,dx.$$

Problem 6.3 A circular pulley is constructed with cross section through its centre as illustrated in Fig. 6.3. Find the volume of the pulley.

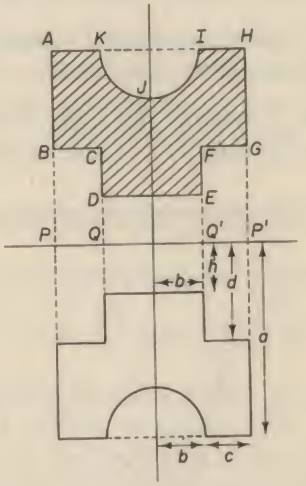

Fig. 6.3

Solution. The pulley can be regarded as being generated by the shaded region being rotation about the x-axis. The volume may be found by subtracting pieces from the cylinder generated by rotating the rectangle $APP'H$ about the x-axis. All the pieces to be subtracted will be cylinders with the exception of the piece generated by the circular groove in the pulley. The volumes of the cylinders are:

$$v(APP'H) = 2\pi a^2(b+c),$$

$$v(DQQ'E) = 2\pi h^2 b,$$

$$v(BPQC) = v(FQ'P'G) = \pi d^2 c.$$

The circular arc KJI has equation $y = a - \sqrt{(b^2-x^2)}$ $(-b < x < b)$ and so the volume generated by rotating the semicircle round the axis, i.e. the volume of the groove in the pulley, is given by the following expression (the required integral is discussed in Problem 5.15):

$$v(KJI) = 2\pi a^2 b - \pi\int_{-b}^{b} (a-\sqrt{(b^2-x^2)})^2\,dx$$

$$= 2\pi a^2 b - \pi\int_{-b}^{b} (a^2 - 2a\sqrt{(b^2-x^2)} + b^2 - x^2)\,dx$$

$$= 2\pi a\int_{-b}^{b}\sqrt{(b^2-x^2)}\,dx - \pi\int_{-b}^{b}(b^2-x^2)\,dx$$

$$= 2\pi a\,.\,\tfrac{1}{2}\pi b^2 - \pi[b^2x - \tfrac{1}{3}x^3]_{-b}^{b} = \pi^2 ab^2 - 4\pi b^3/3.$$

The total volume of the pulley is therefore given by

$$V = 2\pi a^2(b+c) - 2\pi h^2 b - 2\pi d^2 c - \pi^2 ab^2 + 4\pi b^3/3. \qquad \square$$

Readers should choose realistic numerical values for the dimensions of the pulley and obtain a numerical value for V.

Problem 6.4 Find the volume of the solid obtained by rotating the region bounded by the hyperbola $xy = 1$ and the line $x+y = 2\sqrt{2}$ about the line $x = y$.

Solution. We first change the coordinate system so that the line $y = x$

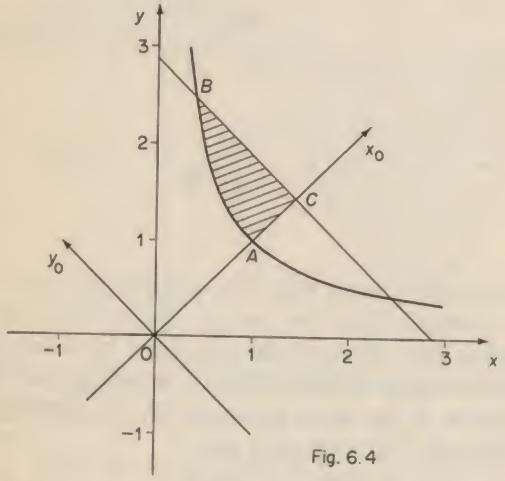

Fig. 6.4

becomes the new x-axis (see Fig. 6.4). The equations which govern the transformation of coordinates are

$$x = (x_0 - y_0)/\sqrt{2}, \qquad y = (x_0 + y_0)/\sqrt{2}.$$

Readers not familiar with this type of transformation will find it discussed in a book in this series, F. Brickell, *Matrices and Vector Spaces.*

The line $x+y = 2\sqrt{2}$ now becomes the line $x_0 = 2$, and the hyperbola

$xy = 1$ becomes $x_0^2 - y_0^2 = 2$. The arc bounding the shaded region in Fig. 6.4 therefore has equation $y_0 = \sqrt{(x_0^2-2)}$. The point A has new coordinates $x_0 = \sqrt{2}$, $y_0 = 0$, and the point C has coordinates $x_0 = 2$, $y_0 = 0$. Thus the volume is given by

$$V = \pi\int_{\sqrt{2}}^{2}(x_0^2-2)\,dx_0 = \pi\left[\frac{1}{3}x_0^3 - 2x_0\right]_{\sqrt{2}}^{2} = \pi\frac{4}{3}(\sqrt{2}-1). \qquad \square$$

There are many solids which are not solids of revolution. Some of these are discussed in L. Marder, *Calculus of Several Variables*, in this series.

6.3 Surfaces of Revolution The formulae used in this and the following application involve the derivative as well as the integral. We make use of the result that the surface obtained by rotating the arc of the graph of the non-negative function $f(x)$ between $x = a$ and $x = b$ about the x-axis has an area given by

$$A = 2\pi\int_a^b f(x)\sqrt{(1+f'(x)^2)}\,dx.$$

If the curve is described parametrically the corresponding formula is

$$A = 2\pi\int_{t_1}^{t_2} y(t)\sqrt{(x'(t)^2 + y'(t)^2)}\,dt.$$

Problem 6.5 A parabolic telescope mirror has diameter across its rim 2m, and a depth of 0·2 m. Find the surface area of the mirror.

Solution. We must first find an algebraic description of the mirror. We may regard it as being formed by rotation about the x-axis of the parabolic arc OP in Fig. 6.5. The measurements tell us that P has coordinates

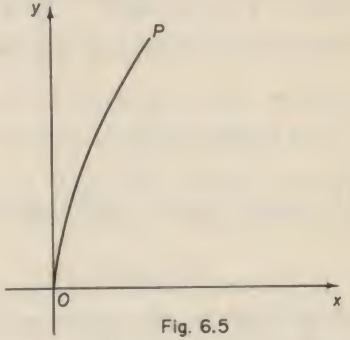

Fig. 6.5

(0·2, 1). Substituting these in the standard form $y^2 = kx$ for the equation of a parabola gives $k = 5$. The arc OP is then represented by the equation

$y = \sqrt{5x}$ $(0 \leqslant x \leqslant 0{\cdot}2)$. Thus $y'(x) = \frac{1}{2}\sqrt{(5/x)}$ and so the surface area of the mirror is given by

$$A = 2\pi \int_0^{0{\cdot}2} \sqrt{5x}\sqrt{(1+5/4x)}\, dx = \pi\sqrt{5} \int_0^{0{\cdot}2} \sqrt{(4x+5)}\, dx$$

$$= \pi\sqrt{5}[\tfrac{2}{3}\tfrac{1}{4}(4x+5)^{\frac{3}{2}}]_0^{0{\cdot}2} = \pi\sqrt{5}(5{\cdot}8^{\frac{3}{2}} - 5^{\frac{3}{2}})/6 \simeq 3{\cdot}25 \text{ m}^2. \quad \square$$

Problem 6.6 Find a formula for the surface area of a torus.

Solution. We must first decide which measurements to use to obtain a result. In this case we shall regard a torus as generated by a circle rotating

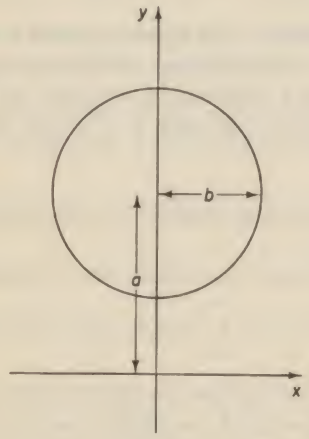

Fig. 6.6

round the x-axis (Fig. 6.6) and use the measurements a and b. The circle may be represented parametrically by means of the equations

$$x = b\cos t, \quad y = a + b\sin t \qquad (0 \leqslant t \leqslant 2\pi).$$

Thus $x'(t) = -b\sin t$ and $y'(t) = b\cos t$, so that the required surface area is given by

$$A = 2\pi \int_0^{2\pi} (a + b\sin t)\sqrt{(b^2\sin^2 t + b^2\cos^2 t)}\, dt$$

$$= 2\pi b \int_0^{2\pi} (a + b\sin t)\, dt = 2\pi b[at - b\cos t]_0^{2\pi} = 4\pi^2 ab. \quad \square$$

6.4 Arc Length In the previous section we have been content to quote formulae. As the first problem in this section we shall derive a formula for arc length, under suitable assumptions.

Problem 6.7 Let $f(x)$ be a function defined on $[a, b]$, continuous on

$[a, b]$ and differentiable on (a, b). Derive a formula for the length of the curve $y = f(x)$.

Solution. We take n points on the curve and join successive points by a line segment (see Fig. 6.7(i)). We make the geometrical assumption that if

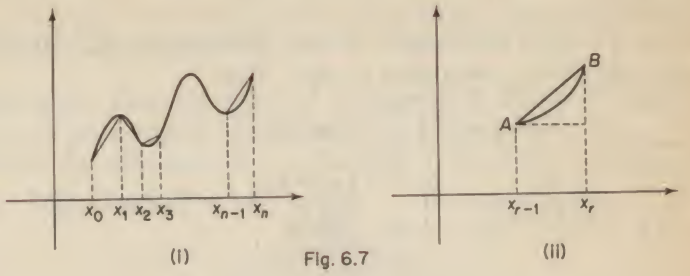

Fig. 6.7

we take a sequence of subdivisions so that the length of the longest such line segment tends to zero, then the sum of the lengths of the line segments tends to a limit, which will be the length of the curve. Referring to Fig. 6.7(ii), we have by the M.V.T.

$$f(x_r) - f(x_{r-1}) = (x_r - x_{r-1})f'(t_r), \qquad x_{r-1} < t_r < x_r.$$

The length l of the line segment AB is given by Pythagoras' Theorem:

$$l^2 = (x_r - x_{r-1})^2 + [f(x_r) - f(x_{r-1})]^2$$

$$= (x_r - x_{r-1})^2 + (x_r - x_{r-1})^2 f'(t_r)^2,$$

$$l = \sqrt{(1 + f'(t_r)^2)}(x_r - x_{r-1}).$$

Thus the total length of the system of line segments is given by

$$L_n = \sum_{r=1}^{n} \sqrt{(1 + f'(t_r)^2)}\,(x_r - x_{r-1}).$$

We now make the analytical assumption that the function

$$g(x) = \sqrt{[1 + f'(x)^2]}$$

has an integral over $[a, b]$. (This will certainly be the case if, for example, $f'(x)$ is continuous.) The work of § 5.7 then tells us that

$$L_n \to \int_a^b \sqrt{[1 + f'(x)^2]}\, dx \qquad \text{as } n \to \infty.$$

Thus the formula for the arc length will be given by

$$L = \int_a^b \sqrt{[1 + f'(x)^2]}\, dx.$$

If the curve is described parametrically then it can be shown that the corresponding formula is

$$L = \int_{t_1}^{t_2} \sqrt{[x'(t)^2 + y'(t)^2]}\, dt. \qquad \square$$

Problem 6.8 Find the length of the curve $y = \ln \sin x$ between $x = \pi/3$ and $x = 2\pi/3$.

Solution. To apply the formula we first calculate the derivative. Using the chain rule, $y'(x) = \cos x / \sin x = \cot x$. Thus

$$L = \int_{\pi/3}^{2\pi/3} \sqrt{(1+\cot^2 x)}\, dx = \int_{\pi/3}^{2\pi/3} \operatorname{cosec} x\, dx$$

$$= \int_{\pi/3}^{2\pi/3} \frac{dx}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} = \int_{\pi/3}^{2\pi/3} \frac{\frac{1}{2}\sec^2 \frac{1}{2}x}{\tan \frac{1}{2}x} dx = [\ln|\tan \tfrac{1}{2}x|]_{\pi/3}^{2\pi/3} = \ln 3. \qquad \square$$

Problem 6.9 Find the length of the curve given parametrically by

$$x = 1 + \tfrac{1}{2}t^2, \quad y = t\cosh t - \sinh t, \qquad 0 \leqslant t \leqslant 1.$$

Solution. The derivatives are given by $x'(t) = t$, $y'(t) = t \sinh t$, and so

$$L = \int_0^1 \sqrt{(t^2 + t^2 \sinh^2 t)}\, dt = \int_0^1 t\sqrt{(1+\sinh^2 t)}\, dt = \int_0^1 t \cosh t\, dt$$

$$= [t \sinh t]_0^1 - \int_0^1 \sinh t\, dt = \sinh 1 - \cosh 1 + \cosh 0 = 1 - 1/e. \qquad \square$$

6.5 Centroids The centroid of a plane region is the point which corresponds to the centre of mass of a uniform lamina of the same shape as the region. Similarly the centroid of a plane curve is the point in the plane corresponding to the centre of mass of a uniform wire with the same shape as the curve.

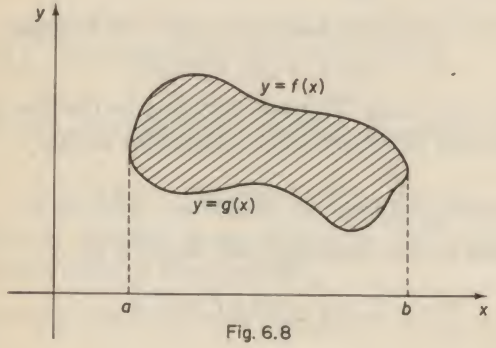

Fig. 6.8

For a region which can be represented by the method illustrated in Fig. 6.8, formulae for the coordinates (X, Y) of the centroid are

$$X = \frac{\int_a^b x[f(x)-g(x)]\, dx}{\int_a^b [f(x)-g(x)]\, dx}, \qquad Y = \frac{\frac{1}{2}\int_a^b [f^2(x)-g^2(x)]\, dx}{\int_a^b [f(x)-g(x)]\, dx},$$

the denominator in each case representing the area of the region. In finding centroids of particular geometrical shapes we make as much use of symmetry as possible.

Problem 6.10 Find the centroid of the region shown in Fig. 6.9, the boundaries being semicircles and portions of the x-axis.

Solution. By symmetry $X = 0$. To find Y, let $f(x) = \sqrt{(4a^2 - x^2)}$ and

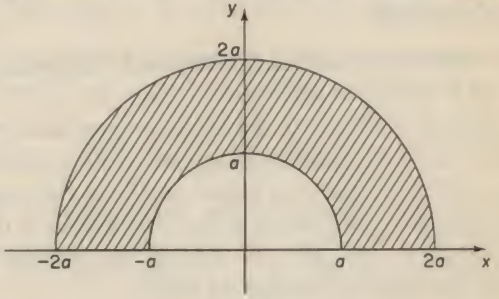

Fig. 6.9

$g(x) = \sqrt{(a^2 - x^2)}$. Using the formula for Y we calculate the integral in three parts to obtain

$$Y = \frac{1}{2A}\int_{-2a}^{-a} f^2(x)\, dx + \frac{1}{2A}\int_{-a}^{a} [f^2(x)-g^2(x)]\, dx + \frac{1}{2A}\int_{a}^{2a} f^2(x)\, dx,$$

where A is the area of the region. From the formula for the area of a circle we have $A = \frac{1}{2}(4\pi a^2 - \pi a^2) = 3\pi a^2/2$. Also

$$\int_{-a}^{a} [f^2(x)-g^2(x)]\, dx = \int_{-a}^{a} [(4a^2-x^2)-(a^2-x^2)]\, dx$$

$$= \int_{-a}^{a} 3a^2\, dx = 6a^3,$$

$$\int_{-2a}^{-a} f^2(x)\, dx = \int_a^{2a} f^2(x)\, dx = \int_a^{2a} (4a^2 - x^2)\, dx = \frac{5}{3}a^3.$$

It follows that

$$Y = \frac{1}{3\pi a^2}\left(6a^3 + \frac{10}{3}a^3\right) = \frac{28a}{9\pi} \simeq 0{\cdot}99a.$$

The centroid is therefore just outside the region, on the axis of symmetry slightly below the point $(0, a)$. $\square$

Because of the similarity of centroid problems to those involving the centre of mass, we shall give no further examples at this point. We shall instead consider the two theorems of Pappus relating centroids to figures of revolution.

Theorems of Pappus (i) Let S be a surface obtained by rotating a plane arc about a coplanar axis which does not intersect the arc. Then the surface area of S is given by the product of the length of the arc and the length of the circle described by its centroid.

(ii) Let V be a solid obtained by rotating a plane region about a coplanar axis not intersecting the region. Then the volume of V is given by the product of the area of the region and the length of the circle described by its centroid.

Problem 6.11 Use the theorems of Pappus to find expressions for the surface area S and volume V of a torus.

Solution. Referring to Fig. 6.6, the torus is generated by rotating the circle about the x-axis. By symmetry the centroid for both the circumference as an arc, and for the disc as a region will lie at the centre. The length of the circle described by this centre is $2\pi a$. The length of the perimeter of the generating circle is $2\pi b$, and the area of the circular region is πb^2. Hence using the theorems of Pappus we have

$$S = 2\pi b \,.\, 2\pi a = 4\pi^2 ab, \qquad V = \pi b^2 \,.\, 2\pi a = 2\pi^2 ab^2. \qquad \square$$

The first result was found by another method in Problem 6.6.

Problem 6.12 Use the theorems of Pappus to find the centroid of a semicircular arc and of a semicircular region.

Solution. We consider a sphere as being generated by a semicircle of radius r rotating about its base. The surface area and volume of the sphere are $4\pi r^2$ and $4\pi r^3/3$ respectively. The arc length and plane area of the semicircle are πr and $\frac{1}{2}\pi r^2$ respectively. Symmetry locates both centroids on the diameter perpendicular to the base of the semicircle. Let a be the distance of the centroid of the arc from the base, and let b be the distance of the centroid of the region from the base. The two theorems of Pappus then give respectively

$$\pi r \,.\, 2\pi a = 4\pi r^2 \text{ and } \tfrac{1}{2}\pi r^2 \,.\, 2\pi b = 4\pi r^3/3, \text{ i.e. } a = 2r/\pi \text{ and } b = 4r/3\pi.$$

$\square$

6.6 Density and Mass It is assumed that the reader has a notion of the physical concept of density, and the mathematical concept of density at a point.

Problem 6.13 A wire has density which differs from that at one end by an amount proportional to the distance from the end. Find an expression for the total mass of the wire.

Solution. Let the length of the wire be l m. Taking the origin as one end of the wire, and measuring x along the wire, the line density $\rho(x)$ is given by $\rho(x) = a+bx$ kg/m. Take a sequence of points $(x_0, x_1, \ldots, x_n)$ along the wire. The segment between x_{r-1} and x_r has mass m_r which satisfies

$$\rho(x_{r-1})(x_r - x_{r-1}) \leqslant m_r \leqslant \rho(x_r)(x_r - x_{r-1})$$

since $\rho(x)$ increases as x increases. The total mass $M = \sum m_r$ therefore satisfies

$$\sum_{r=1}^{n} \rho(x_{r-1})(x_r - x_{r-1}) \leqslant M \leqslant \sum_{r=1}^{n} \rho(x_r)(x_r - x_{r-1}).$$

As we increase the number of points in such a way that $\max(x_r - x_{r-1}) \to 0$ both sums tend to $\int_0^l \rho(x)\,dx$, by § 5.7. Thus we have

$$M = \int_0^l \rho(x)\,dx = \int_0^l (a+bx)\,dx = al + \tfrac{1}{2}bl^2 \text{ kg}. \qquad \square$$

Problem 6.14 Experimental measurements indicate that a planet consists of a spherical core of uniform density 9×10^3 kg/m^3 and radius 1000 km. Between radii 1000 km and 2000 km the density varies in a spherically symmetric fashion, decreasing linearly with the radius from 5×10^3 kg/m^3 to $2{\cdot}5 \times 10^3$ kg/m^3. The atmosphere varies in density in a manner inversely proportional to the fourth power of the distance from

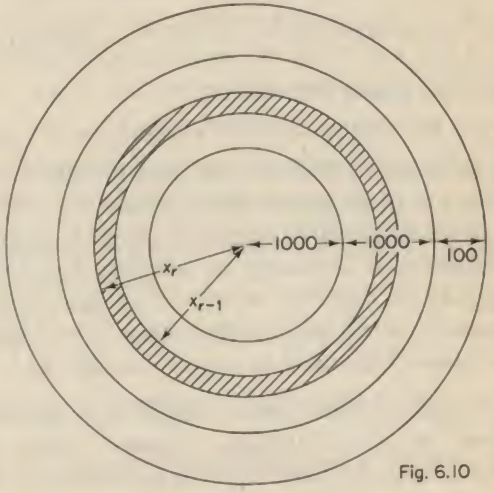

Fig. 6.10

the centre of the planet, the density at the surface being 1 kg/m^3, and it is 100 km thick. Estimate the total mass M of the planet.

Solution. A cross section of the planet is shown in Fig. 6.10. If $\rho(x)$ denotes the density at distance x from the centre then the mass m_r of the spherical shell represented by the shaded portion of the figure satisfies the inequalities

$$4\pi x_{r-1}^2\rho(x_r)(x_r-x_{r-1}) \leqslant m_r \leqslant 4\pi x_r^2\rho(x_{r-1})(x_r-x_{r-1}).$$

Similar reasoning to that in the previous example shows that the total mass contained between radii a and b is $\int_a^b 4\pi r^2\rho(r)\,dr$. The density of the planet's mantle is given by $\rho(r) = a-br$. Since $\rho(1000) = 5\times10^{12}$ kg/km^3 and $\rho(2000) = 2{\cdot}5\times10^{12}$ kg/km^3 we have

$$a = 7{\cdot}5\times10^{12} \text{ and } b = 2{\cdot}5\times10^{9},$$

giving

$$\rho(r) = (7{\cdot}5\times10^{12}-2{\cdot}5\times10^{9}r) \text{ kg/km}^3.$$

The total mass of the mantle is therefore

$$\int_{1000}^{2000} 4\pi r^2(7{\cdot}5\times10^{12}-2{\cdot}5\times10^{9}r)\,dr = 10^{10}\pi\int_{1000}^{2000}(3000r^2-r^3)\,dr$$

$$= \frac{13}{4}\pi\times10^{22} \text{ kg}.$$

The central core, having uniform density $= 9\times10^{12}$ kg/km^3, has mass

$$(4\pi/3)(1000)^3\times9\times10^{12} = 12\pi\times10^{21} \text{ kg}.$$

The atmosphere has density $\rho(r) = cr^{-4}$, and since $\rho(2000) = 10^9$ kg/km^2 this gives $c = 16\times10^{21}$. The mass of the atmosphere is

$$\int_{2000}^{2100} 4\pi r^2 cr^{-4}\,dr = 64\pi\times10^{21}\int_{2000}^{2100} dr/r^2 = (32\pi/21)\times10^{18} \text{ kg}.$$

The total mass of the planet is therefore

$$12\pi\times10^{21}+(13\pi/4)\times10^{22}+(32\pi/21)\times10^{18} \text{ kg}$$

As is to be expected the contribution from the atmosphere is much smaller than that from the rest of the planet. In fact it is of the order of 10^{-4} times the masses of both the core and the mantle. In practice, therefore, we should ignore its contribution in obtaining an estimate for the mass, for the errors introduced by its omission would be no more than those of the experimental measurements. We therefore have $M \simeq 1{\cdot}4\times10^{23}$ kg. (For comparison, the mass of the earth is roughly 6×10^{24} kg.) □

Problem 6.15 A right circular cone of height 10 m and base radius 5 m has density given by $\rho(x) = 100(20-x)$ kg/m^3, where x is the perpendicular distance from the base. Find the total mass M of the cone.

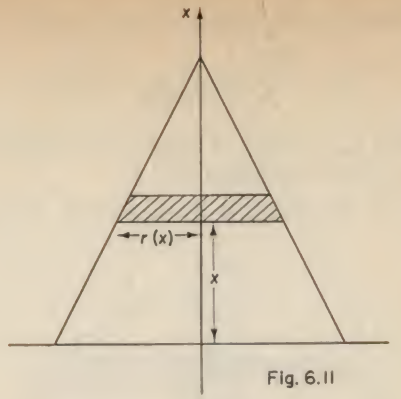

Fig. 6.11

Solution. In Fig. 6.11 the radius at distance x from the base is given by $r(x) = \frac{1}{2}(10-x)$. The mass of the slice of the cone illustrated, of thickness h is approximately

$$h\pi r^2(x)\,\rho(x) = \pi\tfrac{1}{4}(10-x)^2100(20-x)h.$$

Reasoning similar to the previous problems leads to the formula

$$M = (100\pi/4)\int_0^{10}(10-x)^2(20-x)\,dx = 875\,000\pi/6 \simeq 458\,000 \text{ kg}. \quad \square$$

6.7 Centre of Mass

Problem 6.16 Find the centre of mass of the solid cone whose total mass was found in Problem 6.15.

Solution. By symmetry the centre of mass will lie on the axis of the cone, and indeed the resultant moment about this axis is zero. Referring to Fig. 6.11, the mass of the slice of the cone is approximately

$$\pi\tfrac{1}{4}(10-x)^2100(20-x)h$$

and its moment about a diameter of the base is therefore

$$x\pi\tfrac{1}{4}(10-x)^2100(20-x)h$$

A repetition of previous arguments leads to a formula for the total moment about a diameter of the base,

$$T = (100\pi/4)\int_0^{10} x(10-x)^2(20-x)\,dx$$

$$= (100\pi/4)\int_0^{10}(2000x-500x^2+40x^3-x^4)\,dx = 10^6\pi/3$$

From Problem 6.15 the mass M of the cone is $875\,000\pi/6$. The distance X

of the centre of mass from the base is given by $MX = T$, and so

$$X = \frac{10^6\pi}{3}\frac{6}{875\,000\pi} = \frac{16}{7}\text{ m}.$$

Problem 6.17 Find the centre of mass of a uniform spherical cap of height b cut from a sphere of radius $a \geqslant b$.

Solution. Fig. 6.12 represents a cross-section. The mass of the slice

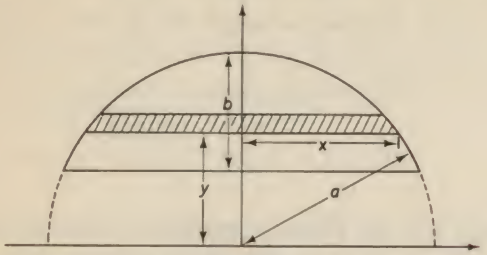

Fig. 6.12

shaded in the figure is approximately $\pi x^2\rho h = \pi\rho(a^2-y^2)h$, where ρ is the uniform density and h is the thickness of the slice. As before this leads to a formula for the total mass M,

$$M = \int_{a-b}^{a} \pi\rho(a^2-y^2)\,dy = \pi\rho b^2(a-\tfrac{1}{3}b).$$

Similarly the total moment T about the x-axis is given by

$$T = \int_{a-b}^{a} \pi y\rho(a^2-y^2)\,dy = \pi\rho b^2(a-\tfrac{1}{2}b)^2.$$

The distance Y of the centre of mass from the origin is now given by $MY = T$, i.e.

$$Y = \frac{(a-\frac{1}{2}b)^2}{(a-\frac{1}{3}b)}.$$

For a hemisphere $b = a$ and the above formula reduces to $Y = 3a/8$. □

Problem 6.18 A composite body is made up of the cone used in Problem 6.15 and a uniform hemisphere of radius 5 m and density 2000 kg/m^3 attached to the base of the cone, the centres coinciding. Find the centre of mass of the body.

Solution. Fig. 6.13 represents a cross-section of the body. The positions of the centre of mass of the separate components are at A and B as shown. The mass of the cone was calculated in Problem 6.15 as $875\,000\pi/6$ kg. The mass of the hemisphere, from Problem 6.17, is

$$\tfrac{2}{3}\pi \times 2000 \times 125 = 10^6\pi/6\text{ kg}.$$

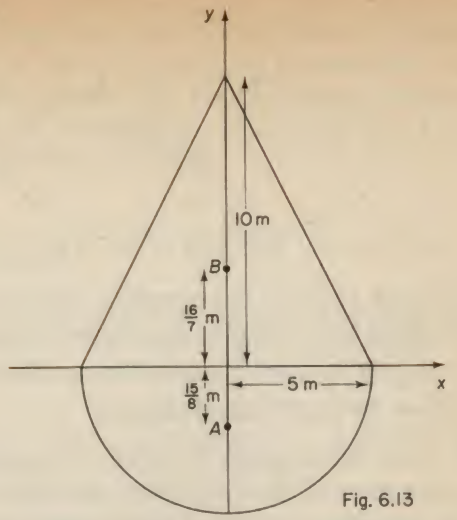

Fig. 6.13

If the distance of the centre of mass of the composite body from the x-axis is Y, then the total moment about the x-axis is

$$\left(\frac{875\,000\pi}{6}+\frac{10^6\pi}{6}\right)Y = \frac{875\,000\pi}{6}\frac{16}{7} - \frac{10^6\pi}{6}\frac{15}{8}$$

which gives $Y = 1/15$ m. □

With a knowledge of the centres of mass of a number of standard geometrical shapes, a large number of compound bodies can be dealt with by the method of Problem 6.18.

6.8 Moments of Inertia The moment of inertia of a particle of mass m about an axis is defined to be md^2, where d is the perpendicular distance between the particle and the axis. Just as in previous examples we dealt with whole bodies in terms of elementary pieces, so we can build up the moment of inertia of a body from subdivisions. Moments of inertia are of importance in the study of the dynamics of rotating bodies. We consider two simple problems first, before quoting two theorems which are then applied further.

Problem 6.19 Find the moment of inertia of a circular wire of line density ρ kg/m, and radius r m about an axis normal to the plane of the wire passing through the centre of the circle.

Solution. In Fig. 6.14 consider the small section of arc cut off by two radii separated by an angle ε. The length of this arc is $r\varepsilon$ and so its mass is $r\rho\varepsilon$. Considering it as a particle it will have a moment of inertia

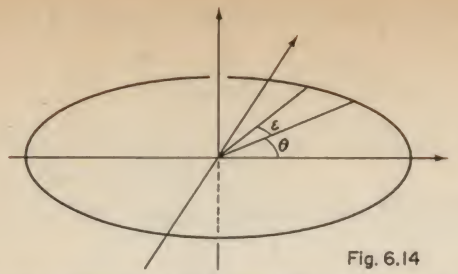

Fig. 6.14

$r\rho\varepsilon . r^2 = \rho r^3 \varepsilon$. If we consider the complete circle divided into a large number of such small arcs then the total moment of inertia is

$$I = \sum \rho r^3 \varepsilon = \rho r^3 \sum \varepsilon = \rho r^3 2\pi = Mr^2 \text{ kg m}^2,$$

where $M = 2\pi r\rho$ is the total mass of the wire. □

Problem 6.20 Find the moment of inertia of a uniform disc about an axis normal to the plane of the disc passing through its centre.

Solution. Let the surface density of the disc be ρ kg/m^2 and its radius a m. The disc is built up by means of concentric annuli as in Fig. 6.15. If

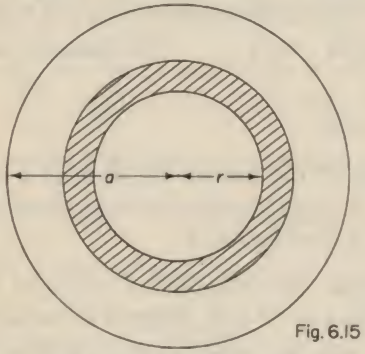

Fig. 6.15

the shaded portion has thickness h then from Problem 6.19 its moment of inertia will be approximately $\rho . 2\pi rh . r^2 = 2\pi\rho r^3 h$. The total moment of inertia of the disc will be

$$I = \int_0^a 2\pi\rho r^3 \, dr = \tfrac{1}{2}\pi\rho a^4.$$

The mass M of the disc is $\pi\rho a^2$ and so $I = \frac{1}{2}Ma^2$ kg m^2. □

Parallel axis theorem Let I be the moment of inertia of a body of mass M about an axis l. Let I_c be the moment of inertia of the body about an axis parallel to l through the centre of mass of the body. Let p be the perpendicular distance between the two axes. Then

$$I = I_c + Mp^2.$$

Perpendicular axes theorem Let l_1 and l_2 be two perpendicular lines lying in the plane of a lamina. Let I_1, I_2 be the moments of inertia of the lamina about l_1 and l_2 respectively. Let l be a line normal to the plane of the lamina through the intersection of l_1 and l_2, and let I be the moment of inertia of the lamina about l. Then

$$I = I_1 + I_2.$$

Problem 6.21 Find the moment of inertia of a uniform disc about a diameter.

Solution. Let M be the mass of the disc and a its radius. By symmetry the moment of inertia is the same about any diameter. Consider two perpendicular diameters and the normal through the centre of the disc. The moment of inertia of the disc about this normal is $\frac{1}{2}Ma^2$ and so by the perpendicular axes theorem the moment of inertia I of the disc about a diameter satisfies $\frac{1}{2}Ma^2 = I + I$, i.e. $I = \frac{1}{4}Ma^2$. □

Problem 6.22 Find the moment of inertia of a uniform spherical cap of height b cut from a sphere of radius $a \geqslant b$ about a line normal to its base passing through a point on the rim.

Solution. We first find the moment of inertia about the axis of symmetry and then use the parallel axis theorem. Referring to Fig. 6.12 the cap is built up from slices in the form of circular discs. The radius of the shaded disc in Fig. 6.12 is $\sqrt{(a^2 - y^2)}$ and so the moment of inertia of the disc is $\frac{1}{2}\pi\rho(a^2 - y^2)^2 h$, where ρ is the uniform density and h the thickness of the disc. The total moment of inertia about the axis of symmetry is then

$$I_c = \int_{a-b}^{a} \tfrac{1}{2}\pi\rho(a^2 - y^2)^2 \, dy = \tfrac{1}{2}\pi\rho b^3 (\tfrac{4}{3}a^2 - ab + \tfrac{1}{5}b^2).$$

From Problem 6.16, the mass M of the cap is $\pi\rho b^2(a - \frac{1}{3}b)$ and so

$$I_c = \frac{Mb(\frac{4}{3}a^2 - ab + \frac{1}{5}b^2)}{2(a - \frac{1}{3}b)}.$$

(For a hemisphere with $b = a$ this becomes $I_c = 2Ma^2/5$.)

To find the moment of inertia I about a line through a point on the rim parallel to the axis of symmetry, we note that the distance between the centre of the base of the cap and a point on the rim is $p = \sqrt{(2ab - b^2)}$. The theorem of parallel axes then gives

$$I = I_c + Mp^2 = Mb\left[\frac{16}{3}a^2 - \frac{13}{3}ab + \frac{13}{15}b^2\right] \Big/ 2\left[a - \frac{1}{3}b\right].$$ □

Problem 6.23 Find the moment of inertia of the cone considered in Problem 6.15 about a diameter of the base.

Solution. We refer to Fig. 6.11. The shaded disc represented in that figure has radius $r(x) = \frac{1}{2}(10-x)$ and mass

$$m = \pi r^2(x)\,\rho(x)h = \pi\tfrac{1}{4}(10-x)^2 100(20-x)h$$

and so by the parallel axis theorem the moment of inertia about a diameter of the base is

$$\frac{100}{16}\pi(10-x)^4(20-x)h + mx^2 = \frac{100}{16}\pi(10-x)^4(20-x)h + \frac{100}{4}\pi(10-x)^2(20-x)x^2h.$$

Thus the total moment of inertia of the complete cone is

$$I = \int_0^{10}\left(\frac{100}{16}\pi(10-x)^4(20-x) + \frac{100}{4}\pi(10-x)^2(20-x)x^2\right)dx$$

The integral to be evaluated is a polynomial, and provided care is taken over the arithmetic there is no difficulty in obtaining $I = 7\times 10^8\pi/64\times 6$. The mass of the cone is $M = 875\,000\pi/6$ and so $I = 25M/2$. □

Problem 6.24 A solid of revolution is formed by rotating the graph $y = f(x)$ $(a \leqslant x \leqslant b)$ of the non-negative function $f(x)$ about the x-axis. The density at any point (x, y, z) of the solid depends only upon x, and is given by $\rho(x)$. Find a formula for the moment of inertia of the solid about the x-axis.

Solution. In Fig. 6.16 the shaded disc has thickness h and radius $f(x)$ at the point x. Its mass is therefore $\pi f^2(x)h\,\rho(x)$ approximately and so its moment of inertia about the x-axis is approximately $\frac{1}{2}\pi f^2(x)h\,\rho(x)\,f^2(x)$.

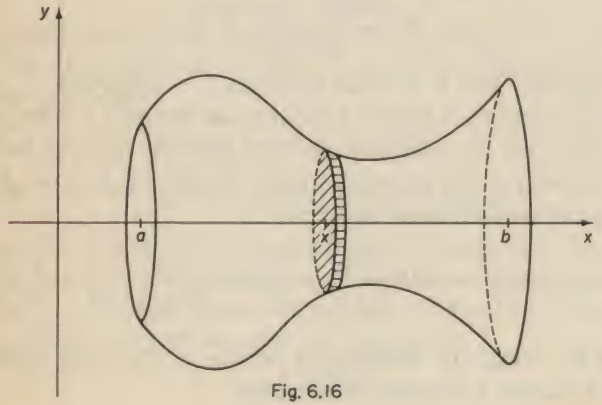

Fig. 6.16

The usual arguments can now be applied to obtain the formula for the moment of inertia of the whole solid about the x-axis,

$$I = \tfrac{1}{2}\pi\int_a^b \rho(x)\,f^4(x)\,dx.$$

This general formula can be applied to any special case. The procedure itself, however, is important as it involves the use of physical and analytical assumptions whose full discussion is outside the scope of this book. Furthermore the procedure itself is simple enough to enable each special case to be considered *ab initio*, rather than remembering the formula. □

6.9 Miscellaneous Problems To end this chapter we present two problems from different areas of physics.

Problem 6.25 An insulating material in the form of a semicircular wire of radius a has charge density $q|\theta|$, where θ is measured as in Fig. 6.17. Find the field at the centre of the semicircle.

Solution. We subdivide the semicircle into small arcs. A typical arc is shown in Fig. 6.17, cut off by two radii with an angle of ε between them.

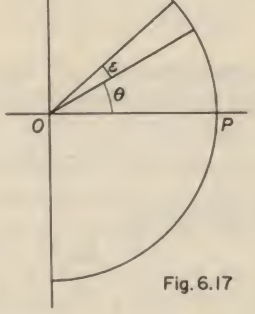

Fig. 6.17

The length of this small arc is $a\varepsilon$ and so the charge on this arc is approximately $q|\theta|a\varepsilon$. The field at O due to this elementary charge is approximately $q|\theta|a\varepsilon/a^2$. By symmetry the resulting field will act along OP, and so we have to sum the components of the elementary charges in this direction. The OP-component of the elementary charge considered is $q|\theta|\cos|\theta|\varepsilon/a$ and so the total field at O will be

$$F = q/a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} |\theta|\cos|\theta|\, d\theta = (2q/a) \int_0^{\frac{1}{2}\pi} \theta \cos\theta\, d\theta$$

$$= (2q/a)\,[\theta \sin\theta]_0^{\frac{1}{2}\pi} - (2q/a) \int_0^{\frac{1}{2}\pi} 1 . \sin\theta\, d\theta$$

$$= (2q/a)(\tfrac{1}{2}\pi - 1). \qquad \square$$

Problem 6.26 A solid metal bar is in the form of a cone of length a m and end radius r m. The metal has density ρ kg/m^3 and specific heat s J/kg °K. The temperature is radially uniform and measured from the apex of the cone obeys the rule $T(x) = ce^{kx}$ °K. Find the total heat content of the bar.

Solution. Consider the elementary disc of thickness h shown in Fig. 6.18. This has radius $y = rx/a$. Its total mass if therefore $\pi\rho r^2x^2h/a^2$ kg.

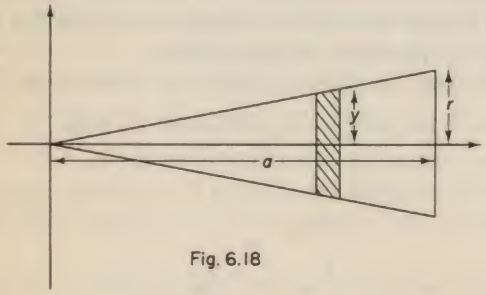

Fig. 6.18

Since its temperature is approximately ce^{kx} throughout, its heat content is $s . ce^{kx} . \pi\rho r^2x^2h/a^2$. Hence the total heat content of the bar is

$$H = \int_0^a (\pi\rho scr^2/a^2)\, x^2e^{kx}\, dx \text{ J}$$

Now integrating by parts twice gives

$$\int_0^a x^2e^{kx}\, dx = \frac{a^2e^{ka}}{k} - \frac{2ae^{ka}}{k^2} + \frac{2e^{ka}}{k^3} - \frac{2}{k^3}.$$

Thus we have

$$H = \frac{\pi\rho scr^2}{k^3a^2}(a^2k^2e^{ka} - 2ake^{ka} + 2e^{ka} - 2) \text{ J} \qquad \square$$

It has only been possible to illustrate some of the applications of integration here, but there are many others in such fields as gravitational attraction, hydrodynamics and statistics.

EXERCISES

1. Find the area of the region enclosed by the curves $y = \cosh x$, $y = \sinh x$ and the lines $x = 1$, $x = -1$.

2. A drainage pipe has a longitudinal section, half of which is illustrated in Fig. 6.19, the broken line representing the axis of the pipe. Find the

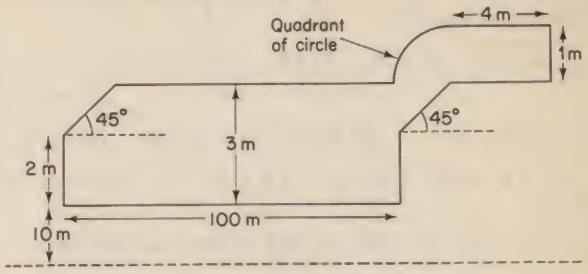

Fig. 6.19

volume of material used in its manufacture, and calculate its total surface area (internal and external).

3. Find a formula for the length of an arc of the parabola $x = at^2$, $y = 2at$, one end of the arc being fixed at the origin.

4. Find the position of the centroid of the region enclosed by the hyperbola $xy = 1$ and the lines $x = 2$, $y = 2$, $x = 0$, $y = 0$. Find its area and hence deduce the volume of the solid obtained by rotating the region about the x-axis.

5. Find the position of the centre of mass of a sector of a circle of which the density is proportional to the distance from the centre.

6. Find the moment of inertia of a uniform rectangular solid of mass M with sides a, b, c about an axis through the centre of mass parallel to the edge of length c. Find also its moment of inertia about one of the edges of length c.

7. Find the moment of inertia of a uniform regular hexagonal prism of length l and side a, and whose mass is M, about the axis of the prism.

8. Find the moment of inertia about its axis of a right circular cylinder whose density measured from one end is $(2+\sin 2\pi x)$ kg/m^3. The length of the cylinder is 10 m and its radius is 2 m.

9. A uniform lamina is constructed from the disc of radius 1 centred at the origin by removing the disc of radius $\frac{1}{2}$ centred at the point $(\frac{1}{2}, 0)$. Find the moment of inertia of the lamina about the x-axis.

Answers to Exercises

Chapter 1

1. $P(x)/Q(x) \to 0$ as $x \to \infty$ if $m < n$,
   $P(x)/Q(x) \to \pm\infty$ as $x \to \infty$ if $m > n$,
   $P(x)/Q(x) \to c$ as $x \to \infty$ if $m = n$, where c is the ratio of the coefficients of the highest powers of x.
2. 26/113. 3. 0. 4. 0. 5. 18. 6. $\frac{1}{2}$.

Chapter 2

1. (i) $f'(x) = -\sin x\, e^{\cos x}$. (ii) $f'(x) = \sec x \tan x$. (iii) $f'(x) = \tanh x$.
2. (i) $f'(x) = 1$ if $x > 0$, $f'(x) = -1$ if $x < 0$; $f(x)$ is not differentiable at $x = 0$.
   (ii) $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ if $x \neq 0$; $f(x)$ is not differentiable at $x = 0$.
3. $f'(x) = 1/\sqrt{(x^2-1)}$.
4. $f'(x) = x^x(1+\ln x)$.
5. $y'(x) = \dfrac{y^2}{x^2}\left(\dfrac{\ln x - 1}{\ln y - 1}\right)$.
6. $f^{(n)}(x) = \dfrac{(-1)^n a^n (2n)!}{2^{2n} n!}(ax+b)^{-(n+\frac{1}{2})}$.
7. $f^{(4)}(x) = 24(-1)^{n+1}(n-5)!\, x^{4-n}$.

Chapter 3

1. $f(x)$ is decreasing in intervals $2n\pi + \pi/6 < x < (2n+1)\pi - \pi/6$ and increasing in the complementary intervals.
2. $f(x)$ has a local maximum at $x = -2$, a local minimum at $x = -\frac{2}{7}$ and a point of inflexion at $x = 1$.
3. Greatest value, $f(1) = 2$, least value $f(2) = 1 + \ln 2$.
4. If the base is of length $2a$ then the required position is $a/\sqrt{3}$ from the base on the axis of symmetry, or at the third vertex, whichever is nearer the base.
5. $y = a(x-1)$.
6. $y = 0$, $x = 0$.
7. See Fig. A.1.
8. See Fig. A.2.
9. The curve is a circle centre $(0, \frac{1}{2})$ radius $\frac{1}{2}$.

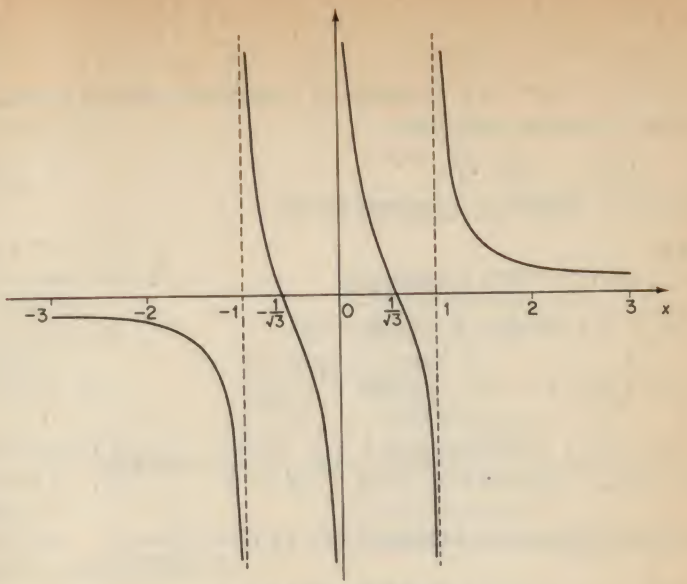

Fig. A.1 Answer to Exercise 3.7

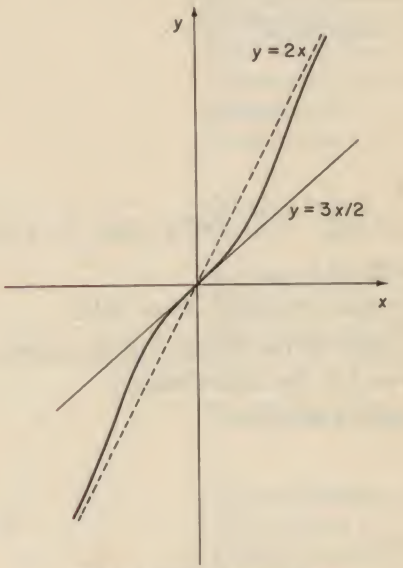

Fig. A.2 Answer to Exercise 3.8

Chapter 4

1. (i) $c = 0, \pi, -\pi$. (ii) $c = 2$.
2. From $f'(x) = nx^{n-1} + p$ a range of possibilities similar to those of Problem 4.4 can be deduced.
3. (i) 1 (ii) 1 (iii) $1/e$.
4. $\sinh(0{\cdot}05) = 0{\cdot}05002$ to 5 decimal places.

Chapter 5

1. (i) $(x^2+3)^5/10$. (ii) $x - \ln|\cos x|$.
   (iii) $\frac{1}{2}x^2 + 7x + 41\ln|x-3| - 12\ln|x-2|$.
   (iv) $3x - \frac{5}{2}\ln|x^2+x+6| - \frac{39}{\sqrt{23}}\tan^{-1}\frac{(2x+1)}{\sqrt{23}}$.
2. (i) $\frac{1}{(a^2-b^2)}\left[\ln\left|\sqrt{\frac{a+b}{a-b}} + \tanh\tfrac{1}{2}x\right| - \ln\left|\sqrt{\frac{a+b}{a-b}} - \tanh\tfrac{1}{2}x\right|\right]$.
   (ii) $(x-1) - 4\sqrt{(x-1)} + 14\ln|1+\sqrt{(x-1)}| + \frac{1}{1+\sqrt{(x-1)}}$.
   (iii) $\frac{1}{4}x(5a^2-2x^2)\sqrt{(a^2-x^2)} + \frac{3}{4}a^4\sin^{-1}(x/a)$.
3. (i) $\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x$ (ii) $(x^2+2)\sinh x - 2x\cosh x$.
   (iii) $\frac{1}{2}(\cos x \sinh x + \sin x \cosh x)$ (iv) $x\sin^{-1}x + \sqrt{(1-x^2)}$.
4. $I = e^{ax}(a\sin bx - b\cos bx)/(a^2+b^2)$,
   $J = e^{ax}(b\sin bx + a\cos bx)/(a^2+b^2)$.
5. $\frac{11}{12} + \frac{1}{2}\ln 2$.
6. $\ln 2$.

Chapter 6

1. $e - 1/e = 2\sinh 1$.
2. $V = 7008 \times \frac{2}{3}\pi + 13\pi^2/2$, $S = 4907\pi + 50\pi\sqrt{2} + 13\pi^2$.
3. $at_0\sqrt{(1+t_0^2)} + a\sinh^{-1}t_0$.
4. $X = Y = 7/4A$, $A = 1 + 2\ln 2$, $V = 7\pi/2$.
5. If the sector has angle 2θ the distance of the centre of mass from the centre is $X = \frac{3}{4}a\sin\theta/\theta$, the radius being a.
6. $M(a^2+b^2)/12$, $M(a^2+b^2)/3$.
7. $5Ma^2/12$.
8. $2M = 160\pi$.
9. $5M/16 = 15\pi\rho/64$.

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